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ANNOTATION

The book contains theoretical studies of the strains in reinforced thin-walled elements (plates, slabs, shells) and their stiffening ribs. The results obtained are mostly presented as tables and graphs.

This book will be useful to scientists, lecturers in strength of materials and theory of elasticity, and designers.

FOREWORD

Thin-walled structures made of thin plates and shells reinforced by thin stiffening ribs are increasingly being used of late.

While these reinforcing elements add only negligibly to the total weight of the structure, they influence considerably its strength, rigidity, and stability. The necessity of designing such structures increased the interest in theoretical and experimental studies of the states of stress and strain of these structural elements (plates and shells reinforced by stiffening ribs).

The first analyses of thin plates and slabs reinforced by straight ribs were performed by G. Bubnov and B.G. Galerkin /6, 19, 20/. The first results in the study of thin plates reinforced around holes are due to S.P. Timoshenko, who applied the methods of strength of materials, based on the theory of curved bars, to the problem of reinforcements around circular and square holes with rounded corners. V.L. Fedorov /139/ was the first to use methods of the theory of elasticity to study the extension of an elastic plane with a circular hole into which an elastic ring is inserted.

Much attention has been paid during the last fifteen years to the effects of stiffening rings, especially after Acad. N.I. Muskhelishvili and his school had developed efficient methods for solving the two-dimensional problem of the theory of elasticity (Kolosov and Muskhelishvili's methods of complex potentials). These methods made it possible not only to formulate this contact problem in its most general form, but also to solve it for many hole shapes.

In view of the mathematical similarity between the two-dimensional problem of the theory of elasticity and the problem of the bending of thin plates, A.I. Lur'e and S.G. Lekhnitskii applied the methods of complex potentials to the problems of bending of isotropic and anisotropic plates.

Large-scale studies of stress concentrations around holes with edges reinforced by elastic ribs began with Mikhlin's paper /77/ on the elastic equilibrium of a circular nonhomogeneous rib consisting of several concentric rings. In papers /114, 115/, Acad. I.N. Mushkelishvili's method was used to solve problems on the stresses in an unbounded plate with a circular hole into which a ring or washer of different material has been welded or inserted under prestressing. These studies were mainly carried out in L'vov and Kiev, and their main results have been published by G.N. Savin /118/, D.V. Vainberg /8/, and M.P. Sheremet'ev /187/.

The present book contains the results of twelve years' studies by the authors and their pupils of the following subjects: 1) analysis of stress concentrations in plates and shells reinforced by stiffening ribs; 2) efficient design of the reinforcing elements, aiming at an optimum structure of thin plates and shells with ribs.

The detailed table of contents eliminates the need to discuss the various chapters in the foreword.

The above-mentioned papers /8, 118, 187/ contain very detailed bibliographies on the many problems discussed in them; a thorough survey of the papers written on this subject was given by Goodyear /34/. References are given where necessary in this book. Here we shall consider the general formulations of the contact problems of reinforcing the edges of holes in thin plates.

In earlier studies the elastic rib was assumed to be wide enough for its state of stress to be described by the equations of the two-dimensional theory of elasticity or the theory of bending of thin plates. In this formulation the problems were solved relatively easily for simple and compound circular ribs. The problem of reinforcing the edges of noncircular holes is far more difficult, and a simplification, yielding a more or less acceptable (although approximate) solution, can be obtained only by greatly restricting the choice of the rib shape. Such a simplification is possible with ribs whose outer and inner contours form coordinate curves.

The stiffening ribs for thin plates whose cross sections have irregular profiles, have uniform or variable cross sections; their elastic behavior is described by the theory of small deformations of thin curved bars. This simplified formulation of the contact problem permitted in many particular cases study of the effects of reinforcing by a circular rib, and its optimum design. With holes shaped like ellipses, squares, triangles with rounded corners, etc., the problem of edge reinforcement by thin elastic ribs is very difficult in this formulation, and could only be solved by successive approximations.

Formulation of the contact problem of reinforcing a thin plate by a sufficiently thin curved rib was further simplified by approximating the boundary conditions. It is thus assumed that the stiffening rib undergoes only tension or compression in the case of the two-dimensional problem, or flexure in its own plane in the case of bending like a thin plate. Using this mathematical model, a solution of the unified contact problem was obtained for the exterior of an elliptic hole whose edge is reinforced by a rib of uniform cross section. This solution, obtained by quadratures, can be generalized to apply to any hole having no corners.

There are thus three ways to formulate contact problems of reinforcing the edges of holes by elastic ribs. The region of application of each variant still remains unknown; it can only be determined accurately on the basis of experimental data and of exact solutions obtained for the combined state of strain of a plate with a hole into which a hollow cylinder has been welded.

The general theory of structures consisting of thin plates and shells reinforced by stiffening ribs should include methods for analyzing the elements of such structures. This would make it possible to take into account the combined state of strain and the interaction between plates (or shells) and thin ribs, permit optimization of reinforcing elements and of their disposition, and enable, e.g., structures of minimum weight to be designed.

This book, which greatly enlarges upon some chapters of /118/ published in 1951, is devoted to a theoretical study of the combined state of strain of thin-walled elements (plates and shells) and of the ribs reinforcing them.

The most important results referring to the interaction between stiffening ribs and plates or shells are given as simple formulas, graphs, and tables. The designer can thus draw the necessary conclusions without entering into the subtleties of the fairly involved theory.

The book contains seven chapters. It has been written in such a way that each chapter can be read independently; this has resulted in some unavoidable repetitions.

Many problems have been incompletely discussed, and some have been barely touched; suitable references are given in those cases.

The authors acknowledge the help of Messrs. Ya. F. Kayuk, L. Ye. Sribnii, L. I. Oshchipko, and R. M. Lutsishin, of the Institute of Mechanics of AN Ukr. SSR and the Chair of General Mechanics and Hydroaeromechanics of L'vov State University, in preparing and arranging the manuscript.

G. N. Savin, N. P. Fleishman

Chapter I

GENERAL THEORY OF BENDING OF THIN PLATES

§ 1. FUNDAMENTAL DIFFERENTIAL EQUATION

Consider the elastic equilibrium of a homogeneous isotropic thin plate of arbitrary contour, subjected to a load $q(x, y)$ normal to its plane. The plate thickness is h = constant. A classification of plates according to the state of stress is given in /94/ and /17/. The fundamentals of the theory of bending of thin plates are found in /19, 61, 63, 64, 80, 90, 94, 136/.

We select the middle plane of the plate as x, y-plane, the Z-axis pointing downward (Figure 1). Let $w(x, y)$ be the deflection, i.e., the Z-component of the vector of displacement of the points lying in the middle plane of the plate being bent.

We shall adopt the following hypotheses of Kirchhoff.

a) Linear elements, perpendicular to the middle plane of the plate before deformation, remain linear and perpendicular to the middle surface of the plate after deformation (hypothesis of straight normals).

b) The stresses σ_Z, normal to the middle surface, are small compared with the two other normal-stress components σ_x and σ_y inside the plate; it can therefore be assumed, when determining σ_x and σ_y, that $\sigma_Z = 0$ everywhere inside the plate (hypothesis of plane state of stress).

In addition, it will be assumed that the plate deflections are small compared with the thickness h, and that the edges can move freely in their plane. This means that the elements of the middle plane of the plate are not extended or compressed during bending. Under these assumptions the x- and y-components $u(x, y, Z)$ and $v(x, y, Z)$ of the vector of displacement of the plate points are (Figure 2):

$$\left.\begin{aligned} u(x, y, Z) &= -Z\frac{\partial w}{\partial x} \\ v(x, y, Z) &= -Z\frac{\partial w}{\partial y} \end{aligned}\right\}, \tag{1.1}$$

where $|Z| \leqslant \frac{h}{2}$ is the distance in the undeformed state of a plate point from the middle plane. Hence, the strain components ε_x, ε_y, and γ_{xy} are:

$$\varepsilon_x = -Z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = -Z\frac{\partial^2 w}{\partial y^2}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2Z\frac{\partial^2 w}{\partial x \partial y}. \tag{1.2}$$

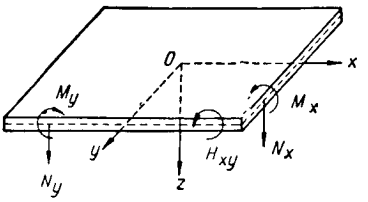


FIGURE 1.

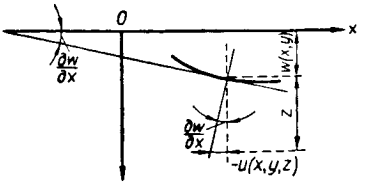


FIGURE 2.

For an isotropic plate the relationships between the stress components σ_x, σ_y, τ_{xy}, τ_{yZ}, τ_{Zx} and the strain components are (for $\sigma_Z = 0$)

$$\left.\begin{aligned} \varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = -Z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = -Z\frac{\partial^2 w}{\partial y^2} \\ \varepsilon_Z = -\frac{\nu}{E}(\sigma_x + \sigma_y) \\ \gamma_{yZ} = \frac{2(1+\nu)}{E}\tau_{yZ}, \quad \gamma_{Zx} = \frac{2(1+\nu)}{E}\tau_{Zx} \\ \gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} = -2Z\frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\}, \tag{1.3}$$

where ν is Poisson's ratio, and E, Young's modulus.

It follows from the first, second, and sixth equations (1.3) that

$$\left.\begin{aligned} \sigma_x = -\frac{EZ}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \\ \sigma_y = -\frac{EZ}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \\ \tau_{xy} = -\frac{EZ}{1+\nu}\cdot\frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\}. \tag{1.4}$$

The stress components must satisfy the equilibrium conditions

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xZ}}{\partial Z} = 0 \\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yZ}}{\partial Z} = 0 \\ \frac{\partial \tau_{Zx}}{\partial x} + \frac{\partial \tau_{Zy}}{\partial y} + \frac{\partial \sigma_Z}{\partial Z} = 0 \end{aligned}\right\}. \tag{1.5}$$

We multiply the first two equations (1.5) by ZdZ and the third by dZ, and integrate over the plate thickness from $-\frac{h}{2}$ to $+\frac{h}{2}$. This yields

$$\left.\begin{aligned} \frac{\partial M_x}{\partial x}+\frac{\partial H_{xy}}{\partial y}-N_x=0 \\ \frac{\partial H_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-N_y=0 \\ \frac{\partial N_x}{\partial x}+\frac{\partial N_y}{\partial y}+q=0 \end{aligned}\right\}, \tag{1.6}$$

where M_x and M_y are the bending moments, H_{xy} is the twisting moment, and N_x and N_y are the shearing forces per unit length of the plate section:

$$\left.\begin{aligned} M_x=\int_{-\frac{h}{2}}^{+\frac{h}{2}}\sigma_x Z dZ=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right) \\ M_y=\int_{-\frac{h}{2}}^{+\frac{h}{2}}\sigma_y Z dZ=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right) \\ H_{xy}=\int_{-\frac{h}{2}}^{+\frac{h}{2}}\tau_{xy} Z dZ=-D(1-\nu)\frac{\partial^2 w}{\partial x\partial y} \end{aligned}\right\}, \tag{1.7}$$

$$\left.\begin{aligned} N_x=\int_{-\frac{h}{2}}^{+\frac{h}{2}}\tau_{xZ} dZ=-D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \\ N_y=\int_{-\frac{h}{2}}^{+\frac{h}{2}}\tau_{yZ} dZ=-D\left(\frac{\partial}{\partial y}\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial x^2}\right) \end{aligned}\right\}, \tag{1.8}$$

where

$$D=\frac{Eh^3}{12(1-\nu^2)} \tag{1.9}$$

is the flexural rigidity of the plate.

Substituting (1.4) in (1.5) and integrating with respect to Z, the boundary conditions being

$$\tau_{xZ}=\tau_{yZ}=0 \quad \text{or} \quad Z=\pm\frac{h}{2},$$

we obtain

$$\left.\begin{aligned}\tau_{xZ} &= E\cdot\frac{4Z^2-h^2}{8(1-\nu^2)}\cdot\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\\ \tau_{yZ} &= E\cdot\frac{4Z^2-h^2}{8(1-\nu^2)}\cdot\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\end{aligned}\right\}. \tag{1.10}$$

Comparison of (1.7) with (1.4) and of (1.8) with (1.10) yields

$$\left.\begin{aligned}\sigma_x &= \frac{12Z}{h^3}M_x\\ \sigma_y &= \frac{12Z}{h^3}M_y\\ \tau_{xy} &= \frac{12Z}{h^3}H_{xy}\\ \tau_{xZ} &= \frac{3(h^2-4Z^2)}{2h^3}N_x\\ \tau_{yZ} &= \frac{3(h^2-4Z^2)}{2h^3}N_y\end{aligned}\right\}, \tag{1.11}$$

It follows from (1.11), (1.7), and (1.8) that the state of stress of the plate is completely defined if its deflection $w(x, y)$ is known.

We derive the differential equation which the function $w(x, y)$ must satisfy by substituting the expressions for N_x and N_y from (1.8) in the third equation (1.6). The result is

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{1}{D}q(x, y), \tag{1.12}$$

or

$$\Delta\Delta w=\frac{q}{D}, \tag{1.13}$$

where $\Delta=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ is the Laplacian and $q=q(x, y)$. This basic differential equation of the theory of bending of thin plates is called the equation of Sophie Germain.

In the particular case $q(x, y)=0$, when the plate is bent only by moments and forces applied at the edges, (1.13) becomes

$$\Delta\Delta w=0, \tag{1.14}$$

i.e., the deflection function $w(x, y)$ is biharmonic.

Airy's stress function in the two-dimensional theory of elasticity is also biharmonic. This similarity in the mathematical formulation of these problems made it possible to apply the powerful method of Muskhelishvili to solve problems in the theory of bending of thin plates /10, 63, 65, 82/.

The problem of the bending of a thin plate is thus reduced to integrating (1.13) or (1.14) with boundary conditions depending on the manner of supporting the plate edges.

Polar coordinates (r, θ) are used for problems of circular and annular plates, r being the distance of the point considered from the pole (the origin of the cartesian coordinates), and θ, the polar angle, measured clockwise from the x-axis.

In this system (1.13) becomes

$$\Delta\Delta w = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial w}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 w}{\partial\theta^2}\right) = \frac{q}{D}. \tag{1.15}$$

The moments (1.7) and shearing forces (1.8), expressed in polar coordinates (r, θ) (cf., e. g., /136/, Chapter VII): are

$$\left.\begin{aligned} M_r &= -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\cdot\frac{\partial w}{\partial r} + \frac{\nu}{r^2}\cdot\frac{\partial^2 w}{\partial\theta^2}\right)\\ M_\theta &= -D\left(\frac{1}{r}\cdot\frac{\partial w}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 w}{\partial\theta^2} + \nu\frac{\partial^2 w}{\partial r^2}\right)\\ H_{r\theta} &= -(1-\nu)D\left(\frac{1}{r}\cdot\frac{\partial^2 w}{\partial r\partial\theta} - \frac{1}{r^2}\cdot\frac{\partial w}{\partial\theta}\right) \end{aligned}\right\}, \tag{1.16}$$

$$N_r = -D\frac{\partial}{\partial r}(\Delta w), \qquad N_\theta = -D\frac{1}{r}\cdot\frac{\partial}{\partial\theta}(\Delta w), \tag{1.17}$$

where the Laplacian Δ is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial\theta^2}. \tag{1.18}$$

In the case of axisymmetrical bending of circular or annular plates, the deflection function is independent of the polar angle θ, i. e., $w = w(r)$. Equations (1.16) and (1.17) then become

$$\left.\begin{aligned} M_r &= -D\left(\frac{d^2 w}{dr^2} + \frac{\nu}{r}\cdot\frac{dw}{dr}\right)\\ M_t &= -D\left(\frac{1}{r}\cdot\frac{dw}{dr} + \nu\frac{d^2 w}{dr^2}\right)\\ H_{r\theta} &= 0, \qquad N_\theta = 0\\ N_r &= -D\frac{d}{dr}(\Delta w) \end{aligned}\right\}, \tag{1.19}$$

where

$$\Delta = \frac{d^2}{dr^2} + \frac{1}{r}\cdot\frac{d}{dr}. \tag{1.20}$$

§ 2. COMPLEX REPRESENTATION OF THE GENERAL SOLUTION. DEGREE OF DETERMINACY OF THE FUNCTIONS $\varphi(z)$ AND $\psi(z)$

The general integral of (1.13) is written in the form of the sum

$$w(x, y) = w^1(x, y) + w^0(x, y), \tag{1.21}$$

where $w^1(x, y) = w^1$ is the general integral of the biharmonic equation (1.14), and $w^0(x, y) = w^0$ is any single-valued particular solution of (1.13).

Applying Muskhelishvili's method /82/, we represent the biharmonic function $w^1(x, y)$, according to Goursat's formula, by two analytical functions $\varphi(z)$ and $\chi(z)$ of the complex variable $z = x+iy$:

$$w^1 = 2\,\mathrm{Re}\,[\bar{z}\varphi(z) + \chi(z)]. \tag{1.22}$$

Substitution of (1.22) in (1.21) yields the general integral of (1.13) in the form

$$w(x, y) = 2\,\mathrm{Re}\,[\bar{z}\varphi(z) + \chi(z)] + w^0(x, y). \tag{1.23}$$

The bending moments (1.7) and shearing forces (1.8) are with $\psi(z) = \dfrac{d\chi(z)}{dz}$ /63/:

$$\left.\begin{aligned}
M_x &= M_x^0 - D\{2(1+\nu)[\varphi'(z) + \overline{\varphi'(z)}] + (1-\nu)[\bar{z}\varphi''(z) + \\
&\qquad + z\overline{\varphi''(z)} + \psi'(z) + \overline{\psi'(z)}]\} \\
M_y &= M_y^0 - D\{2(1+\nu)[\varphi'(z) + \overline{\varphi'(z)}] - (1-\nu)[\bar{z}\varphi''(z) + \\
&\qquad + z\overline{\varphi''(z)} + \psi'(z) + \overline{\psi'(z)}]\} \\
H_{xy} &= -iD(1-\nu)[\bar{z}\varphi''(z) + \psi'(z) - z\overline{\varphi''(z)} - \overline{\psi'(z)}] + H_{xy}^0 \\
N_x &= -8D\,\mathrm{Re}\,[\varphi''(z)] + N_x^0 \\
N_y &= -8D\,\mathrm{Re}\,[i\varphi''(z)] + N_y^0
\end{aligned}\right\}, \tag{1.24}$$

where $M_x^0, M_y^0, H_{xy}^0, N_x^0, N_y^0$ are single-valued, being given by (1.7) and (1.8) when w is replaced by w^0.

We introduce the following complex combinations of these magnitudes:

$$\left.\begin{aligned}
M_y - M_x + 2iH_{xy} &= 4D(1-\nu)[\bar{z}\varphi''(z) + \psi'(z)] + \\
&\qquad + (M_y^0 - M_x^0 + 2iH_{xy}^0) \\
M_x + M_y &= -4D(1+\nu)[\varphi'(z) + \overline{\varphi'(z)}] + (M_x^0 + M_y^0) \\
N_x - iN_y &= -8D\varphi''(z) + (N_x^0 - iN_y^0)
\end{aligned}\right\}. \tag{1.25}$$

We also rewrite (1.1) in the form

$$\begin{aligned}
u + iv &= -\left(\frac{\partial w^1}{\partial x} + i\frac{\partial w^1}{\partial y}\right)Z - \left(\frac{\partial w^0}{\partial x} + i\frac{\partial w^0}{\partial y}\right)Z = \\
&= -2[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}]Z - \left(\frac{\partial w^0}{\partial x} + i\frac{\partial w^0}{\partial y}\right)Z.
\end{aligned} \tag{1.26}$$

It follows from (1.25) that for a given state of stress in the plate, i. e., for known values of M_x, M_y, and H_{xy}, the functions $\varphi(z)$ and $\psi(z)$ are fully determined except for expressions of the form

$$iCz + \alpha^* + i\beta^* \text{ and } \alpha^{*\prime} + i\beta^{*\prime}, \text{ respectively} \qquad (1.27)$$

where C, α^*, β^*, $\alpha^{*\prime}$, $\beta^{*\prime}$ are real constants. It is thus possible to add the expressions $iCz + \alpha^* + i\beta^*$ to the function $\varphi(z)$ and $\alpha^{*\prime} + i\beta^{*\prime}$ to the function $\psi(z)$ without affecting the state of stress in the plate.

Thus, in a complex simply-connected domain S, with the origin of coordinates at any point of the domain, we can require that

$$I_m[\varphi'(0)] = 0, \quad \varphi(0) = 0, \quad \psi(0) = 0, \qquad (1.28)$$

where I_m is the imaginary part of the expression in brackets.

Let the domain be infinite (for instance, an unbounded plate with a hole) and the origin of coordinates lie outside the domain S, i. e., inside the hole. For finite stresses at infinity, (1.28) can then be written

$$I_m[\varphi'(\infty)] = 0, \quad \varphi(\infty) = 0, \quad \psi(\infty) = 0. \qquad (1.29)$$

Conditions (1.28) or (1.29) eliminate completely the indeterminancy of the functions $\varphi(z)$ and $\psi(z)$ when the state of stress in the plate is given.

Setting $w^0 = 0$, we substitute in (1.23) and (1.26) the functions $\varphi(z) = iCz + \alpha^* + i\beta^*$ and $\psi(z) = \alpha^{*\prime} + i\beta^{*\prime}$ or $\chi(z) = (\alpha^{*\prime} + i\beta^{*\prime})z + \alpha^{*\prime\prime} + i\beta^{*\prime\prime}$, where $\alpha^{*\prime\prime}$ and $\beta^{*\prime\prime}$ are real constants. The components of the displacement vector of the plate points are then

$$\left.\begin{aligned} u(x, y) &= -2(\alpha^* + \alpha^{*\prime})Z \\ v(x, y) &= -2(\beta^* - \beta^{*\prime})Z \\ w(x, y) &= 2(\alpha^* + \alpha^{*\prime})x + 2(\beta^* - \beta^{*\prime})y + 2\alpha^{*\prime\prime} \end{aligned}\right\}. \qquad (1.30)$$

These expressions define a rigid-body displacement of the plate, as ought to be expected.

We shall now determine how much the indeterminacy of the functions $\varphi(z)$ and $\psi(z)$ is reduced by specifying the plate deflections. The state of stress in the plate is then completely determined, so that the functions $\varphi(z)$ and $\psi(z)$ are again obtained except for expressions (1.27) which must not affect the plate deflections.

It therefore follows from (1.30) that

$$\alpha^* + \alpha^{*\prime} = 0, \quad \beta^* - \beta^{*\prime} = 0, \quad \alpha^{*\prime\prime} = 0. \qquad (1.31)$$

Thus, when the deflections are given only the constants C and $\beta^{*\prime\prime}$, and one of the pairs (α^*, β^*) or $(\alpha^{*\prime}, \beta^{*\prime})$, are arbitrary.

§ 3. GENERAL FORM OF THE FUNCTIONS $\varphi(z)$, $\psi(z)$, AND $\chi(z)$

If the domain of the plate is simply connected, the functions $\varphi(z)$ and $\psi(z)$ are single-valued in it, but if the domain is multiply-connected, the

functions $\varphi(z)$ and $\psi(z)$ may be multiple-valued. We are interested in the case when the plate is weakened by N holes. If the plate dimensions are finite, we have a multiply-connected domain delimited by $N+1$ simple closed contours; one of them (L_{N+1}) is the external contour of the plate, while the other $L_j (j=1, 2, \ldots, N)$ are the contours of the holes (Figure 3). It is assumed that these contours have no common points. We shall determine the form of the functions $\varphi(z)$ and $\psi(z)$ in the case of a multiply-connected domain /82, pp. 120-124/. The origin of coordinates is placed at the center of one hole. It follows from the second equation (1.25) that the real part of $\varphi'(z)$ is single-valued. The imaginary part of this function increases by $2\pi i A_k$, for each clockwise pass along any closed curve L_k' surrounding one of the contours $L_k (k = 1, 2, \ldots, N)$, where A_k is a real constant. Such an increment will obviously be obtained if we take the function in the form

$$\varphi'(z) = \varphi_*'(z) + \sum_{k=1}^{N} A_k \ln(z - z_k), \tag{1.32}$$

where $z_1, z_2, \ldots, z_N$ are arbitrary points inside the contours $L_1, L_2, \ldots, L_N$, and $\varphi_*'(z)$ is a single-valued function, holomorphic in the multiply-connected domain S.

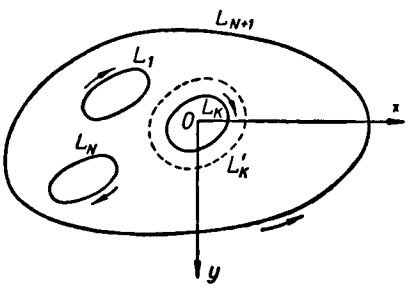


FIGURE 3.

Integration of (1.32) yields

$$\varphi(z) = \sum_{k=1}^{N} A_k z \ln(z - z_k) + \sum_{k=1}^{N} (\alpha_k + i\beta_k) \ln(z - z_k) + \varphi^*(z), \tag{1.33}$$

where α_k and β_k are real constants, and $\varphi^*(z)$ is a new single-valued function. It follows from the first equation (1.25) that $\psi'(z)$ is a single-valued function. Therefore $\psi(z) = \int \psi'(z)\,dz$ becomes

$$\psi(z) = \sum_{k=1}^{N} (\alpha_k' + i\beta_k') \ln(z - z_k) + \psi^*(z), \tag{1.34}$$

where α_k' and β_k' are real constants, and $\psi^*(z)$ is a holomorphic single-valued function. The function $\chi(z) = \int \psi(z)\,dz$ has the form

$$\chi(z) = \sum_{k=1}^{N} [(\alpha_k' + i\beta_k') z + (\alpha_k'' + i\beta_k'')] \ln(z - z_k) + \chi^*(z), \tag{1.35}$$

where α_k' and β_k' are real constants, and $\chi^*(z)$ is a holomorphic single-valued function.

The displacement components u and v are assumed to be single-valued functions. It therefore follows from (1.26) that

$$\{u + iv\}_{L_k'} = -\{\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}\}_{L_k'} Z = 0, \tag{1.36}$$

where the sign $\{\ \}_{L_k'}$ denotes the increment of the expression between braces for each complete clockwise pass along any curve L_k'. Substitution of the functions $\varphi(z)$ and $\psi(z)$ found above yields

$$\alpha_k - \alpha_k' = 0, \qquad \beta_k + \beta_k' = 0. \tag{1.37}$$

In addition, the deflections $w(x, y)$ given by (1.23) must be single-valued:

$$\{w(x, y)\}_{L_k'} = \{\bar{z}\varphi(z) + \chi(z) + z\overline{\varphi(z)} + \overline{\chi(z)}\}_{L_k'} = 0. \tag{1.38}$$

Substituting (1.33) and (1.35) in (1.38), and taking (1.37) into account, we obtain

$$\beta_k^* = 0. \tag{1.39}$$

We conclude by virtue of (1.37) and (1.39) that the functions $\varphi(z)$, $\psi(z)$, and $\chi(z)$ must have the form

$$\left.\begin{aligned} \varphi(z) &= \sum_{k=1}^{N} A_k z \ln(z - z_k) + \sum_{k=1}^{N} a_k \ln(z - z_k) + \varphi^*(z) \\ \psi(z) &= \sum_{k=1}^{N} \bar{a}_k \ln(z - z_k) + \psi^*(z) \\ \chi(z) &= \sum_{k=1}^{N} (\bar{a}_k z + \alpha_k^*) \ln(z - z_k) + \chi^*(z) \end{aligned}\right\}, \tag{1.40}$$

where $a_k = \alpha_k + i\beta_k$. The functions $\varphi^*(z)$, $\psi^*(z)$, $\chi^*(z)$ are single-valued and continuous in the plate domain except possibly at the boundary and at infinity.

§ 4. PRINCIPAL VECTOR AND PRINCIPAL MOMENT

We shall determine the principal vector $\vec{P}$ and the principal moment $\vec{M}$ of the forces applied to the contour L' of part S^* assumed cut out of the plate and bounded by the contours L' and L'' (Figure 4).

We assume that no transverse load acts: $q(x, y) = w^0(x, y) = 0$, and that the part considered is subjected only to stresses X_n, Y_n, Z_n, acting on it along L', due to the remainder of the plate;

$$\left.\begin{aligned} X_n &= \sigma_x \cos(n, x) + \tau_{xy} \cos(n, y) + \tau_{xz} \cos(n, Z) \\ Y_n &= \tau_{yx} \cos(n, x) + \sigma_y \cos(n, y) + \tau_{yz} \cos(n, Z) \\ Z_n &= \tau_{zx} \cos(n, x) + \tau_{zy} \cos(n, y) + \sigma_z \cos(n, Z) \end{aligned}\right\}. \tag{1.41}$$

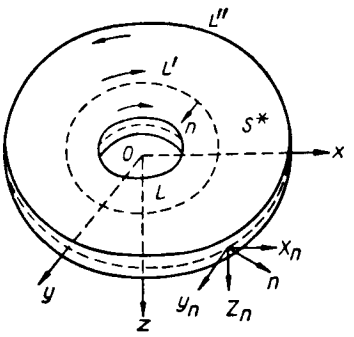


FIGURE 4.

Since $\cos(n,Z) = 0$ and $\sigma_z = 0$ we obtain by virtue of (1.11):

$$\left.\begin{aligned} X_n &= \frac{12Z}{h^3}[M_x \cos(n, x) + H_{xy} \cos(n, y)] \\ Y_n &= \frac{12Z}{h^3}[H_{xy} \cos(n, x) + M_y \cos(n, y)] \\ Z_n &= \frac{3(h^2 - 4Z^2)}{2h^3}[N_x \cos(n, x) + N_y \cos(n, y)] \end{aligned}\right\}, \tag{1.42}$$

where n is the outer normal relative to S^*, and

$$\cos(n, x) = -\frac{dy}{ds}, \qquad \cos(n, y) = \frac{dx}{ds}. \tag{1.43}$$

The principal vector of the forces X_n, Y_n, Z_n applied to the contour L' is directed along the Z-axis:

$$P_z^* = \int_{L'} \int_{-\frac{h}{2}}^{+\frac{h}{2}} Z_n dZ ds. \tag{1.44}$$

The projections on the coordinate axes of the principal moment (about the origin of coordinates) of the forces X_n, Y_n, Z_n are

$$\left.\begin{aligned} M_x^* &= \int_{L'} \int_{-\frac{h}{2}}^{+\frac{h}{2}} (Z_n y - Y_n Z)\, dZ ds \\ M_y^* &= \int_{L'} \int_{-\frac{h}{2}}^{+\frac{h}{2}} (X_n Z - Z_n x)\, dZ ds \end{aligned}\right\}. \tag{1.45}$$

Substituting in (1.44) and (1.45) for X_n, Y_n, Z_n from (1.42), we obtain

$$\left.\begin{aligned} P_z^* &= \int_{L'} (N_x dy - N_y dx) \\ M_x^* &= \int_{L'} (M_y - yN_y)\, dx + (N_x y - H_{xy})\, dy \\ M_y^* &= \int_{L'} (N_y x - H_{xy})\, dx + (M_x - N_x x)\, dy \end{aligned}\right\}. \tag{1.46}$$

The principal moment is thus

$$M_x^* + iM_y^* = \int_{L'} [(M_y + izN_y - iH_{xy})\, dx + (iM_x - H_{xy} - izN_x)\, dy. \tag{1.47}$$

Substituting (1.24) in (1.46) and (1.47), and integrating along the contour L', we obtain

$$P_z^* = 4iD\,\{\varphi'(z) - \overline{\varphi'(z)}\}_{L'}, \tag{1.48}$$

$$\begin{aligned} M_x^* + iM_y^* = 2D\,\{&(1-\nu)\,[\overline{\psi(z)} + z\overline{\varphi'(z)}] - (3+\nu)\,\varphi(z) - \\ &- 2z\,[\overline{\varphi'(z)} - \varphi'(z)]\}_{L'}. \end{aligned} \tag{1.49}$$

It is assumed that the curve L' has no common point with the hole contour L.

We conclude that P_z^*, defined by (1.48), and $M_x^* + iM_y^*$, defined by (1.49), are, respectively, the principal vector and the principal moment of the forces applied to the hole contour, if integration along L' is in a clockwise direction.

§ 5. FORM OF THE FUNCTIONS $\varphi(z)$, $\psi(z)$, AND $\chi(z)$ FOR A MULTIPLY-CONNECTED DOMAIN

We shall express the constants A_k and $a_k = \alpha_k + i\beta_k$, entering in the functions $\varphi(z)$ and $\psi(z)$ in (1.40), through the principal vector P_{zk}^* and the principal moment $M_{xk}^* + iM_{yk}^*$ of the forces applied to the contour L_k of a hole in the plate.

Substituting (1.40) in (1.48) and (1.49) and taking into account the direction of integration, we obtain

$$A_k = \frac{1}{2\pi i} \cdot \frac{iP_{zk}^*}{8D}, \qquad a_k = \frac{1}{2\pi i} \frac{M_{xk}^* + iM_{yk}^*}{8D}. \tag{1.50}$$

Thus, the functions $\varphi(z)$, $\psi(z)$ and $\chi(z)$ become

$$\left.\begin{aligned}
\varphi(z) &= \frac{1}{2\pi i}\sum_{k=1}^{N}\left[\frac{iP_{zk}^{*}}{8D}\cdot z+\frac{M_{xk}^{*}+iM_{yk}^{*}}{8D}\right]\ln(z-z_k)+\varphi^{*}(z)\\
\psi(z) &= -\frac{1}{2\pi i}\sum_{k=1}^{N}\frac{1}{8D}(M_{xk}^{*}-iM_{yk}^{*})\ln(z-z_k)+\psi^{*}(z)\\
\chi(z) &= \frac{1}{2\pi i}\sum_{k=1}^{N}\left[i\alpha_{*k}^{*}-z\frac{M_k^{*}-iM_k^{*}}{8D}\right]\ln(z-z_k)+\chi^{*}(z)
\end{aligned}\right\}, \qquad (1.51)$$

where $\varphi^{*}(z)$, $\psi^{*}(z)$ and $\chi^{*}(z)$ are single-valued and continuous in the domain of the plate.

It can be shown (/82/, (p.127) that for a homogeneous* state of stress at infinity in an unbounded plate, we obtain from (1.51)

$$\left.\begin{aligned}
\varphi(z) &= \frac{1}{2\pi i}\sum_{k=1}^{N}\frac{1}{8D}(M_{xk}^{*}+iM_{yk}^{*})\ln(z-z_k)+(B_0+iC_0)z+\varphi^{0}(z)\\
\psi(z) &= -\frac{1}{2\pi i}\sum_{k=1}^{N}\frac{1}{8D}(M_{xk}^{*}-iM_{yk}^{*})\ln(z-z_k)+(B'+iC')z+\psi^{0}(z)
\end{aligned}\right\}, \qquad (1.52)$$

where B_0, C_0, B', C' are real constants, and $\varphi^{0}(z)$ and $\psi^{0}(z)$ are functions which for sufficiently large $|z|$ can be expanded in the form

$$\varphi^{0}(z)=\sum_{k=0}^{\infty}\frac{c_k}{z^k}, \qquad \psi^{0}(z)=\sum_{k=0}^{\infty}\frac{c_k'}{z^k}. \qquad (1.53)$$

In addition, we must have in this case:

$$P_z^{*}=0, \qquad (1.54)$$

where P_z^{*} is the principal vector of the shearing forces applied to all holes in the plate.

It is seen from (1.51) that if the load applied to the contours of the holes is statically equivalent to zero (the principal vector and the principal moments vanish), or if no external forces at all are applied at these holes, the functions $\varphi(z)$ and $\psi(z)$ will be single-valued and continuous in the entire domain S of the plate.

The solution of the problem for a multiply-connected, in particular doubly-connected, domain (bounded plate with hole) is exceedingly cumbersome. In practice, the dimensions of holes which are very near to the outer edge of the plate are often small compared to the dimensions of the plate. In such cases the plate is considered as unbounded and having

* i.e., the moments M_x, M_y, and H_{xy} are finite at infinity;

$$M_x^{(\infty)}=\text{const}, \quad M_y^{(\infty)}=\text{const}, \quad H_{xy}^{(\infty)}=\text{const}.$$

a hole, it being assumed that the state of stress at the outer plate edge is not affected by the hole.

§ 6. BASIC PROBLEMS. BOUNDARY CONDITIONS

We shall now consider the boundary conditions for a thin plate whose middle plane forms the domain S, which is bounded by the curved smooth contour $L = \sum_{j=1}^{N+1} L_j$.

1) The plate edge L is free of geometrical constraints and is loaded by external bending moments $m(s)$ and shearing forces $p(s)$, where s is the length of the arc measured on L from an arbitrary point. We have then the following two conditions:

$$M_n = m(s), \qquad Q_n^* = N_n + \frac{\partial H_{n\tau}}{\partial s} = p(s) \ (\text{on } L), \tag{1.55}$$

where M_n, $H_{n\tau}$ and N_n are, respectively, the bending moment, the twisting moment, and the shearing force per unit length of plate contour, in the section whose outer normal is n. The magnitude Q_n^* is called generalized shearing force.

The magnitudes M_n, $H_{n\tau}$ and N_n are expressed through the moments M_x, M_y, H_{xy}, and the forces N_x, N_y by the standard transformation formulas, similar to the transformation formulas for the stress components when the coordinate axes are rotated:

$$\left.\begin{aligned} M_n &= M_x \cos^2(n,x) + M_y \cos^2(n,y) + 2H_{xy}\cos(n,x)\cos(n,y) \\ H_{n\tau} &= (M_y - M_x)\cos(n,x)\cos(n,y) + H_{xy}[\cos^2(n,x) - \cos^2(n,y)] \\ N_n &= \pm[N_x \cos(n,x) + N_y \cos(n,y)] \end{aligned}\right\}. \tag{1.56}$$

Here n is the outer normal to L; the plus sign is taken on the outer contour of the plate, and the minus sign on the contour of the hole in the plate.

The boundary conditions (1.55) can be expressed through the deflection function $w(x, y)$ by substituting (1.56) in (1.55) and expressing the moments and forces $M_x, M_y, \ldots, N_y$ by (1.7) and (1.8). We obtain finally

$$\left.\begin{aligned} m(s) &= -D\left\{\nu\Delta w + (1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\cos^2(n,x) + \frac{\partial^2 w}{\partial y^2}\sin^2(n,x) + \right.\right. \\ &\qquad \left.\left. + 2\frac{\partial^2 w}{\partial x \partial y}\sin(n,x)\cos(n,x)\right]\right\} \\ p(s) &= D\left\{\frac{\partial \Delta w}{\partial n} + (1-\nu)\frac{\partial}{\partial s}\left[\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right)\sin(n,x)\cos(n,x) + \right.\right. \\ &\qquad \left.\left. + \frac{\partial^2 w}{\partial x \partial y}\cos^2(n,x) - \frac{\partial^2 w}{\partial x \partial y}\sin^2(n,x)\right]\right\} \end{aligned}\right\}. \tag{1.57}$$

Following Lekhnitskii /63/, we shall express the boundary conditions (1.55) through the functions $\varphi(z)$ and $\psi(z)$. Starting from an arbitrary point

on the contour L, we integrate the second condition (1.55) along the arc s. This yields

$$P + H_{n\tau} = f(s) + C, \tag{1.58}$$

$$P = \int_0^s N_n ds, \qquad f(s) = \int_0^s p(s)\, ds, \tag{1.59}$$

where C is a real constant.

Substituting (1.56) in (1.55) and (1.58), we obtain

$$\left.\begin{aligned} &M_x \cos^2(n, x) + M_y \cos^2(n, y) + 2H_{xy} \cos(n, x) \cos(n, y) = m(s) \\ &(M_y - M_x) \cos(n, x) \cos(n, y) + H_{xy} [\cos^2(n, x) - \cos^2(n, y)] + \\ &\qquad + P = f(s) + C \end{aligned}\right\}. \tag{1.60}$$

We can find equivalent simpler conditions from (1.60). Multiplying the first equation (1.60) by $\cos(n, y)$, the second equation by $\cos(n, x)$, and adding the two, we obtain

$$M_y \cos(n, y) + (H_{xy} + P) \cos(n, x) = m \cos(n, y) + (f + C) \cos(n, x). \tag{1.61}$$

By virtue of (1.43), we can write (1.60) and (1.61) in the form*

$$\left.\begin{aligned} &M_x dy - (H_{xy} - P)\, dx = m dy + (f + C)\, dx \\ &(H_{xy} + P)\, dy - M_y dx = -m dx + (f + C)\, dy \end{aligned}\right\}. \tag{1.62}$$

Contour conditions of the type (1.62) can be expressed through the functions $\varphi(z)$ and $\psi(z)$. We first substitute in (1.62) for M_x, M_y, and H_{xy} their expressions from (1.24), and for P its value

$$\begin{aligned} P = \int_0^s (N_x dy - N_y dx) &= 4iD \int_0^s \{[\varphi''(z) - \overline{\varphi''(z)}]\, dx + \\ &+ [\varphi''(z) + \overline{\varphi''(z)}]\, i dy\} + \int_0^s (N_x^0 dy - N_y^0 dx) = \\ &= 4iD [\varphi'(z) - \overline{\varphi'(z)}] + \int_0^s (N_x^0 dy - N_y^0 dx). \end{aligned} \tag{1.63}$$

We then multiply the first equation (1.62) by $-i$, and add it to the second. Integrating along the contour s, we obtain the following final boundary condition on L for the first basic problem /63, p. 206/:

$$\begin{aligned} (1 - \nu) [z\overline{\varphi'(z)} + \overline{\psi(z)} - (3 + \nu) \varphi(z)] &= -\frac{1}{2D} \int_0^s (m^0 + if^0)(dx + i dy) + \\ &+ \frac{1}{2D} \int_0^s (m + if)(dx + i dy) + iC_0 z + C_* \end{aligned} \tag{1.64}$$

* [The first expression is obtained by multiplying the first equation (1.60) by $\cos(n, x)$, the second equation (1.60) by $\cos(n, y)$ and subtracting the latter from the former.]

or, expressed in the conjugated magnitudes,

$$-\varkappa\overline{\varphi(z)}+\bar{z}\varphi'(z)+\psi(z)=\frac{1}{2D(1-\nu)}[l^0(s)-l(s)]-iC_1\bar{z}+C_2, \tag{1.65}$$

where

$$l(s)=-\int_0^s(m-if)(dx-idy)=$$

$$=-\int_0^s\left[m(s_1)-i\int_0^{s_1}p(s_2)\,ds_2\right](dx-idy), \tag{1.66}$$

$$l^0(s)=-\int_0^s(m^0-if^0)(dx-idy)=$$

$$=-\int_0^s\left[m^0(s_1)-i\int_0^{s_1}p^0(s_2)\,ds_2\right](dx-idy), \tag{1.67}$$

$\varkappa=\frac{3+\nu}{1-\nu}$; C_1 and C_2 are, respectively, real and complex constants; ν is Poisson's ratio; s, s_1, s_2 are arcs measured along L; $m^0(s_1)$ and $p^0(s_2)$ are obtained by replacing w by w^0 in (1.57).

When the state of stress in the plate is given, the function $\varphi(z)$ is determined except for the expression $iCz+\alpha^*+i\beta^*$, and the function $\psi(z)$, except for an expression of the form $\alpha^{*\prime}+i\beta^{*\prime}$ (cf. (1.27)). We can utilize this indeterminacy of the functions $\varphi(z)$ and $\psi(z)$, and specify on one contour L_1 the value of C_2 in (1.65). On the remaining contours this constant will have different values, to be determined when solving the problem. The constant C_1 is determined from the condition that the displacements be single-valued.

If the domain of the plate is simply-connected, we can set $C_2=0$ in (1.65). In addition, when the domain is bounded, we can prescribe arbitrarily C_1 and $\varphi(0)$ or $\psi(0)$ (it is assumed that the origin of coordinates lies inside the domain S). We thus set

$$\left.\begin{matrix}\varphi(0)=0 \quad \text{or} \quad \psi(0)=0\\ C_1=0\end{matrix}\right\}, \tag{1.68}$$

For an unbounded domain with finite stresses at infinity, we can write instead

$$\begin{gathered}\operatorname{Im}\varphi'(\infty)=0,\\ \varphi(\infty)=0 \quad \text{or} \quad \psi(\infty)=0.\end{gathered} \tag{1.69}$$

2) The deflections $w(s)$ and the slopes of the middle surface with respect to the xy-plane are specified along the contour L of a plate forming a simply-connected domain, i.e., the normal derivative $\frac{\partial w}{\partial n}$ is given as a function of the arc length s (n is the outer normal). In particular, if $w(s)$ and $\frac{\partial w}{\partial n}$ vanish on L, the plate edge is built in at a fixed support. In this case the boundary conditions are

$$w=w_*(s), \qquad \frac{\partial w}{\partial n}=\frac{\partial w_*}{\partial n} \text{ on } L, \tag{1.70}$$

where $w_*(s)$ and $\frac{\partial w_*}{\partial n}$ are known functions on L.

We can write (1.70) in a more convenient form. If $w(s)$ is known on L, its derivative

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} \tag{1.71}$$

is also known. In addition

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial n} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial n}, \tag{1.72}$$

where (Figure 5)

$$-\frac{\partial y}{\partial s} = \frac{\partial x}{\partial n} = \cos(n, x) = \cos\alpha; \quad \frac{\partial x}{\partial s} = \frac{\partial y}{\partial n} = \cos(n, y) = \sin\alpha, \tag{1.73}$$

α is the angle between the outer normal n and the x-axis.

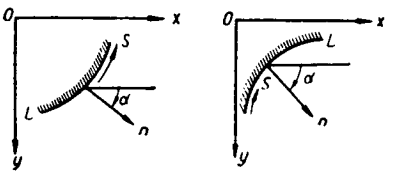


FIGURE 5.

Thus,

$$\frac{\partial w}{\partial n} + i\frac{\partial w}{\partial s} = i\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right)\left(\frac{\partial x}{\partial s} + i\frac{\partial y}{\partial s}\right) = \\ = e^{i\alpha}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right). \tag{1.74}$$

Hence, according to (1.70),

$$\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y} = e^{-i\alpha}\left(\frac{\partial w_*}{\partial n} + i\frac{\partial w_*}{\partial s}\right) \text{ on } L. \tag{1.75}$$

Inserting (1.23) into the left-hand side of (1.75), we obtain

$$\frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) = \overline{\varphi(z)} + \bar{z}\varphi'(z) + \psi(z) + \frac{1}{2}\left(\frac{\partial w^0}{\partial x} - i\frac{\partial w^0}{\partial y}\right). \tag{1.76}$$

Hence, the boundary condition of the second basic problem is:

$$\overline{\varphi(z)} + \bar{z}\varphi'(z) + \psi(z) = \frac{e^{-i\alpha}}{2}\left(\frac{\partial w_*}{\partial n} + i\frac{\partial w_*}{\partial s}\right) - \frac{1}{2}\left(\frac{\partial w^0}{\partial x} - i\frac{\partial w^0}{\partial y}\right). \tag{1.77}$$

Chapter II

BENDING OF THIN PLATES REINFORCED BY CURVED STIFFENING RIBS

§ 1. STATEMENT OF THE PROBLEM

Although structures consisting of thin plates reinforced by annular stiffening ribs have been used for a long time in engineering practice, the problem of analyzing such plates has not been solved completely so far. To study the influence of stiffening ribs on the strength and rigidity of thin plates, we consider /148, 152/ a thin elastic isotropic plate whose middle surface forms a bounded multiply-connected domain S in the plane $z = x + iy$. The boundary L of S consists of a set of simple closed curves $L_j (j = 1, 2, \ldots, m+1)$, with the curve L_{m+1} enclosing all others (Figure 6). The plate is reinforced by annular stiffening ribs made of a different material. The axial lines of the ribs are denoted by $\gamma_2, \ldots, \gamma_l$. The line $\Gamma = \sum_{k=1}^{l} \gamma_k$ divides the domain S into $l+1$ domains so that $S = S_0 + S_1 + \ldots + S_l$. We assume for simplicity that the domains $S_k (k = 1, 2, \ldots, l)$ are simply connected, while the domain S_0 is multiply-connected and bounded by the set of curves Γ and $L = \sum_{j=1}^{m+1} L_j$.

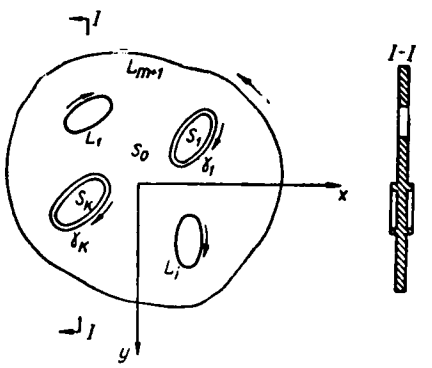


FIGURE 6.

We assume further that the boundaries L and Γ neither touch nor intersect each other. A given transverse load (distributed load $q_k(x, y)$, concentrated moments and forces) is applied to the plate in domains

Unlike the right-hand side of (1.65) the right-hand side of (1.77) contains no arbitrary magnitudes. Since the displacements of all points of the plate are completely determined in the case considered, it follows from (1.31) that of the five constants entering in (1.27) we can only prescribe C and one of the pairs (α^*, β^*) or $(\alpha^{*\prime}, \beta^{*\prime})$. If the origin of coordinates lies in the domain of the plate, we can set either

$$\mathrm{Im}[\varphi'(0)] = 0, \quad \varphi(0) = 0. \tag{1.78}$$

or

$$\mathrm{Im}[\varphi'(0)] = 0, \quad \psi(0) = 0. \tag{1.79}$$

For an unbounded domain with finite stresses at infinity, we can set either

$$\mathrm{Im}[\varphi'(\infty)] = 0, \quad \varphi(\infty) = 0 \tag{1.80}$$

or

$$\mathrm{Im}[\varphi'(\infty)] = 0, \quad \psi(\infty) = 0. \tag{1.81}$$

In fact, (1.77) is equivalent to

$$\frac{\partial w}{\partial n} + i\frac{\partial w}{\partial s} = \frac{\partial w_*}{\partial n} + i\frac{\partial w_*}{\partial s} \quad \text{on } L, \tag{1.82}$$

and not to (1.70).

If the plate forms a simply-connected domain, rigid-body displacement of its edge L is unimportant. We can then replace (1.70) by (1.77), since under varying conditions the plate deflections will not differ, the entire plate only being displaced as a rigid body.

If the plate forms a multiply-connected domain, (1.70) cannot be replaced by (1.77) in general.

3) The plate edge is simply supported. The deflection and the bending moment at L vanish, i.e.,

$$w = 0, \quad M_n = 0 \quad \text{on } L, \tag{1.83}$$

where M_n is defined by (1.56).

The boundary conditions (1.83) are homogeneous; generally speaking, however, nonhomogeneous boundary conditions are also possible. Thus, if external bending moments $m(s)$ are applied to the hinged support of a plate, we have instead of (1.83):

$$w = 0, \quad M_n = m(s) \text{ on } L. \tag{1.84}$$

We shall in the next chapter derive the boundary conditions for a more general case, when the plate edge is connected to a thin elastic bar which is deformed together with the plate (cf. § 16 of Chapter II).

$S_k (k=0, 1, 2, \ldots, l)$, while external bending moments $m(s)$ and forces $p(s)$ act on the boundary L (conditions of the first basic problem):

$$m(s) = M_n(s), \quad p(s) = N_n(s) + \frac{\partial H_{n\tau}(s)}{\partial s} \text{ on } L. \tag{2.1}$$

Alternatively, the conditions of the second basic problem are:

$$w = w_*(s), \quad \frac{\partial w}{\partial n} = \frac{\partial w_*}{\partial n} \text{ on } L. \tag{2.2}$$

The stiffening ribs are considered as thin elastic rings possessing constant flexural and torsional rigidities. The behavior of such ribs is known from the theory of small deformations of thin curved bars /66, 87, 102/.

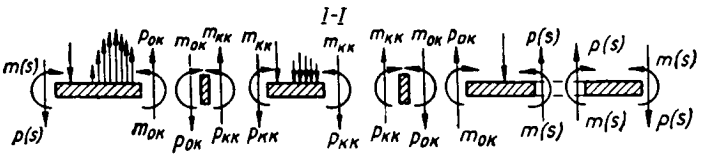


FIGURE 7.

We assume further that one of the principal axes of inertia of the cross section of each stiffening rib lies in the xy-plane. This principal axis of inertia coincides with the principal normal to the axial line γ_k ($k = 1, 2, \ldots, l$) of the ring considered.

Since the width of the stiffening ribs is small, we assume that the domains S_0 and S_k touch along the curves γ_k.

The bending moments and forces, acting on the k-th rib from the side of the domain S_0, will be denoted by $m_{0k}(s)$ and $p_{0k}(s)$, and those acting on this rib from the side of the domain S_k, by $m_{kk}(s)$ and $p_{kk}(s)$ ($k = 1, 2, \ldots, l$), where s is the arc of the contour γ_k, measured clockwise from an arbitrary point. If a load acts directly on the stiffening rib, i.e., along γ_k, it can, by virtue of the above assumptions, be considered as applied to one of the adjacent domains S_0 or S_k. The positive directions of forces and moments are shown in Figure 7, which represents a section of the plate along the line I—I (cf. Figure 6).

The k-th rib is thus subjected to the total load

$$\left.\begin{aligned} m_k(s) &= m_{0k}(s) - m_{kk}(s) \\ p_k(s) &= p_{0k}(s) - p_{kk}(s) \end{aligned}\right\}. \tag{2.3}$$

In other words, the bending moments and the shearing forces undergo (so far unknown) jumps $m_k(s)$ and $p_k(s)$ during transition from S_0 to S_k.

Under the action of the load (2.3) the rib is deformed together with the plate near it. The continuity condition for the deflections and slopes on the contour γ_k yields

$$w_0 = w_k = \Delta_k, \tag{2.4}$$

$$\frac{\partial w_0}{\partial n} = \frac{\partial w_k}{\partial n} = \theta_{\tau k}, \tag{2.5}$$

where w_0 is the plate deflection in S_0, w_k is the deflection in $S_k (k = 1, 2, \ldots, l)$, Δ_k is the deflection of the points on the axis of the k-th rib, n is the outer normal relative to S_0, $\theta_{\tau k}$ is the angle of torsion of the elastic line of the rib, i.e., the angle of rotation of the normal rib section about the tangent τ to the contour γ_k. The tangent lies in the direction of increasing values of s (Figure 6).

Differentiating (2.4) with respect to s, we can write the condition of strain continuity in complex form*

$$\left.\begin{aligned} &\frac{\partial w_0}{\partial n} + i\frac{\partial w_0}{\partial s} = \frac{\partial w_k}{\partial n} + i\frac{\partial w_k}{\partial s} \quad \text{on } \gamma_k \\ &\frac{\partial w_k}{\partial n} + i\frac{\partial w_k}{\partial s} = \theta_{\tau k} - i\theta_{nk} \quad \text{on } \gamma_k \end{aligned}\right\}, \tag{2.6}$$

where $\theta_{nk} = -\frac{d\Delta_k}{ds}$ is the angle of bending of the rib, i.e., the angle of rotation of the tangent to the rib axis about the normal in the xy-plane at the given point of the axis. The angles $\theta_{\tau k}$ and θ_{nk} are considered as positive when the rotation is anticlockwise.

The boundary conditions (2.6) can be transformed, similarly to (1.74), by writing

$$\frac{\partial w}{\partial n} + i\frac{\partial w}{\partial s} = it\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right),$$

where $t = x + iy$ is the complex coordinate of the point considered of $\Gamma\left(\dot{t} = \frac{dt}{ds}\right)$. Since $\dot{t}\,\bar{\dot{t}} = 1$, we can rewrite (2.6) in the form

$$\frac{\partial w_k}{\partial x} - i\frac{\partial w_k}{\partial y} = \frac{\bar{\dot{t}}}{i}(\theta_{\tau k} - i\theta_{nk}) \quad \text{on } \gamma_k, \tag{2.7}$$

$$\frac{\partial w_0}{\partial x} - i\frac{\partial w_0}{\partial y} = \frac{\partial w_k}{\partial x} - i\frac{\partial w_k}{\partial y} \quad \text{on } \gamma_k. \tag{2.8}$$

Thus, (2.3), (2.7), and (2.8) must be satisfied on γ_k. The right-hand side of (2.7) contains the unknown strains of rib γ_k, due to load (2.3). In addition, the deflections $w_0(x, y)$ must also satisfy the boundary conditions (2.1) or (2.2) on the external boundary L of S_0. Different boundary conditions can also be specified on L.

§ 2. TRANSFORMING THE BOUNDARY CONDITION (2.7)

We transform the boundary condition (2.7) by considering the equilibrium and deformation of the thin elastic ring $\gamma_k (k = 1, 2, \ldots, l)$, acted upon by the loads $p_k(s)$ and $m_k(s)$ ((2.3)).

* The replacement of boundary conditions (2.4) and (2.5) by (2.6) is permissible in this case (cf. § 6 of Chapter I).

We denote by

$$\vec{\theta}_k(s) = \theta_{\tau k}\vec{\tau} + \theta_{nk}\vec{n} + \theta_{bk}\vec{b}$$

the vector of rotation of the trihedron, formed by the basis vectors $\vec{\tau}$, $\vec{n}$, $\vec{b}$, during deformation of the ring. The binormal unit vector $\vec{b}$ is directed downward parallel to the Z-axis. We also introduce the vector

$$\vec{\varepsilon}_k(s) = \delta\omega_{\tau k}\vec{\tau} + \delta\omega_{nk}\vec{n} + \delta\omega_{bk}\vec{b},$$

where $\delta\omega_{nk}$, $\delta\omega_{bk}$, and $\delta\omega_{\tau k}$ are the increments of the principal components of the curvature and torsion of the ring.

The vectors $\vec{\theta}_k(s)$ and $\vec{\varepsilon}_k(s)$ are connected by the Clebsch relationships /66/:

$$\left.\begin{aligned}
\delta\omega_{\tau k} &= \frac{d\theta_{\tau k}}{ds} + \omega_{nk}\theta_{bk} - \omega_{bk}\theta_{nk} \\
\delta\omega_{nk} &= \frac{d\theta_{nk}}{ds} + \omega_{bk}\theta_{\tau k} - \omega_{\tau k}\theta_{bk} \\
\delta\omega_{bk} &= \frac{d\theta_{bk}}{ds} + \omega_{\tau k}\theta_{nk} - \omega_{nk}\theta_{\tau k}
\end{aligned}\right\}, \qquad (2.9)$$

where ω_{nk}, ω_{bk}, and $\omega_{\tau k}$ are the principal components of the curvature and torsion of the ring.

The components of the principal moment

$$\vec{L}_k = L_{\tau k}\vec{\tau} + L_{nk}\vec{n} + L_{bk}\vec{b}$$

of the internal forces acting in section s of the ring, are related to the components of $\vec{\varepsilon}_k(s)$ by the Kirchhoff equations

$$\delta\omega_{\tau k} = \frac{L_{\tau k}}{C_k}, \quad \delta\omega_{nk} = \frac{L_{nk}}{A_k}, \quad \delta\omega_{bk} = \frac{L_{bk}}{B_k}, \qquad (2.10)$$

where L_{nk} and L_{bk} are the bending moments [in fact, the dimensions of L are force × length!]; $L_{\tau k}$ is the twisting moment; A_k and B_k are the flexural rigidities of the ring about the $\vec{n}$ and $\vec{b}$ axes, respectively; C_k is the torsional ridigity.

The equilibrium equations of a ring element are

$$\frac{d\vec{V}_k}{ds} + \vec{p}_k = 0, \quad \frac{d\vec{L}_k}{ds} + (\vec{\tau} \times \vec{V}_k) + \vec{m}_k = 0, \qquad (2.11)$$

where $\vec{V}_k$ is the principal vector of the internal forces acting in section s of the ring, $\vec{p}_k$ and $\vec{m}_k$ are, respectively, the external force and bending moment applied to unit length of the axis of the ring element considered. In the case considered we obtain from (2.3);

$$\vec{p}_k = p_k(s)\vec{b}, \quad \vec{m}_k = m_k(s)\vec{\tau}. \qquad (2.12)$$

In addition,

$$\omega_{\tau k} = 0, \quad \omega_{nk} = 0; \quad \omega_{bk} = \frac{1}{\varrho_k}, \tag{2.13}$$

since each stiffening rib has a constant cross section, and its axis γ_k is a plane curve with variable radius of curvature ϱ_k. The first two equations (2.9) become, when (2.13) is inserted:

$$\left.\begin{aligned} \frac{d\theta_{\tau k}}{ds} &= \delta\omega_{\tau k} + \theta_{nk} \cdot \frac{1}{\varrho_k} \\ \frac{d\theta_{nk}}{ds} &= \delta\omega_{nk} - \theta_{\tau k} \cdot \frac{1}{\varrho_k} \end{aligned}\right\}. \tag{2.14}$$

This becomes in complex form:

$$\frac{d}{ds}(\theta_{\tau k} - i\theta_{nk}) = \delta\omega_{\tau k} - i\delta\omega_{nk} + \frac{i}{\varrho_k}(\theta_{\tau k} - i\theta_{nk}). \tag{2.15}$$

Multiplying both sides of (2.15) by $\dot{\bar{t}}ds$ and integrating, we obtain

$$\int_0^s \frac{d}{ds}(\theta_{\tau k} - i\theta_{nk})\,\dot{\bar{t}}ds =$$

$$= \int_0^s (\delta\omega_{\tau k} - i\delta\omega_{nk})\,\dot{\bar{t}}ds + i\int_0^s \frac{1}{\varrho_k}(\theta_{\tau k} - i\theta_{nk})\,\dot{\bar{t}}ds.$$

Integrating the left-hand side by parts, taking into account that

$$\frac{d\dot{\bar{t}}}{ds} = \frac{d}{ds}(\dot{x} - i\dot{y}) = \ddot{x} - i\ddot{y} = -\frac{\dot{y}}{\varrho_k} - i\frac{\dot{x}}{\varrho_k} = -\frac{i\dot{\bar{t}}}{\varrho_k}, \tag{2.16}$$

we find

$$(\theta_{\tau k} - i\theta_{nk})\,\dot{\bar{t}} = \int_0^s (\delta\omega_{\tau k} - i\delta\omega_{nk})\,\dot{\bar{t}}ds + C_{3k}, \tag{2.17}$$

where $C_{3k} = (\theta^0_{\tau k} - i\theta^0_{nk})\,\dot{\bar{t}}_0$ is a complex constant. Substituting (2.10) in (2.17), we obtain

$$(\theta_{\tau k} - i\theta_{nk})\,\dot{\bar{t}} = \int_0^s \left(\frac{L_{\tau k}}{C_k} - i\frac{L_{nk}}{A_k}\right)\dot{\bar{t}}ds + C_{3k}. \tag{2.18}$$

We express the integral in the right-hand side of (2.18) through the load (2.12) on the rib by projecting the second equation (2.11) on the moving axes $\vec{\tau}$ and $\vec{n}$. Since

$$\frac{d\vec{\tau}}{ds} = \frac{\vec{n}}{\varrho_k}, \quad \frac{d\vec{n}}{ds} = -\frac{\vec{\tau}}{\varrho_k}, \quad \frac{d\vec{b}}{ds} = 0,$$

we obtain

$$\left.\begin{aligned}\frac{dL_{\tau k}}{ds} &= \frac{L_{nk}}{\varrho_k} - m_k(s)\\ \frac{dL_{nk}}{ds} &= -\frac{L_{\tau k}}{\varrho_k} + V_{bk}\end{aligned}\right\}, \tag{2.19}$$

where V_{bk} is the shearing force acting in section s of the rib.

Multiplying by $\dot{\bar{t}}ds$ both sides of the first equation (2.19) and integrating yields

$$\int_0^s \frac{dL_{\tau k}}{ds}\dot{\bar{t}}ds = \int_0^s \frac{L_{nk}}{\varrho_k}\dot{\bar{t}}ds - \int_0^s m_k(s)\dot{\bar{t}}ds.$$

Integrating the left-hand side by parts, using (2.16) and noting that $\dot{t}\,\dot{\bar{t}} = 1$, we obtain

$$L_{\tau k} = -i\dot{t}\int_0^s L_{\tau k}\frac{\dot{\bar{t}}}{\varrho_k}ds + \dot{t}\int_0^s\left[\frac{L_{nk}}{\varrho_k} - m_k(s)\right]\dot{\bar{t}}ds + C_k'\dot{t}, \tag{2.20}$$

where C_k' is a complex constant. Proceeding in the same way with the second equation (2.19), we obtain

$$L_{nk} = -i\dot{t}\int_0^s L_{nk}\frac{\dot{\bar{t}}}{\varrho_k}ds + \dot{t}\int_0^s\left[V_{bk} - \frac{L_{\tau k}}{\varrho_k}\right]\dot{\bar{t}}ds + C_k''\dot{t}, \tag{2.21}$$

where C_k'' is also a complex constant. Multiplying (2.21) by i and subtracting it from (2.20) yields then

$$L_{\tau k} - iL_{nk} = -\dot{t}\int_0^s [m_k(s) + iV_{bk}]\dot{\bar{t}}ds + C_{4k}\dot{t}, \tag{2.22}$$

where $C_{4k} = (L_{\tau k}^0 - iL_{nk}^0)\dot{\bar{t}}_0$ is a new complex constant. Integrating the first equation (2.11) and using (2.12), we obtain

$$V_{bk} = -\int_0^s p_k(s)\,ds + V_{bk}^0, \tag{2.23}$$

where V_{bk}^0 is a real constant. Inserting (2.23) in (2.22) yields

$$L_{\tau k} - iL_{nk} = -\dot{t}\int_0^s\left[m_k(s_1) - i\int_0^{s_1} p_k(s_2)\,ds_2\right]\dot{\bar{t}}ds_1 - i\dot{t}V_{bk}^0(\bar{t} - \bar{t}_0) + C_{4k}\dot{t}, \tag{2.24}$$

where t_0 is the coordinate of $s = 0$.

We write, by analogy with (1.66),

$$\left.\begin{aligned}I_k &= I_k(s) = I_k(t) = -\int_0^s\left[m_k(s_1) - i\int_0^{s_1} p_k(s_2)\,ds_2\right]\dot{\bar{t}}ds_1\\ I_k^*(s) &= \dot{t}[I_k(s) - iV_{bk}^0(\bar{t} - \bar{t}_0) + C_{4k}]\end{aligned}\right\}: \tag{2.25}$$

The unknown functions $l_k(t)$ $(k = 1, 2, \ldots, l)$ depend obviously on the jump in the loads (2.3) across γ_k.

Using (2.25), we write (2.24) in the form

$$L_{\tau k} - iL_{nk} = l_k^*(s).$$

Hence

$$\left.\begin{aligned} L_{\tau k} &= \frac{1}{2}[l_k^*(s) + \overline{l_k^*(s)}] \\ L_{nk} &= \frac{1}{2i}[\overline{l_k^*(s)} - l_k^*(s)] \end{aligned}\right\}. \tag{2.26}$$

Substituting (2.26) in (2.18), we obtain

$$\dot{i}(\theta_{\tau k} - i\theta_{nk}) = \frac{1}{2}\int_0^s J_k(s)\,ds + C_{3k}, \tag{2.27}$$

where

$$J_k(s) = \left[l_k^*(s)\left(\frac{1}{C_k} + \frac{1}{A_k}\right) + \overline{l_k^*(s)}\left(\frac{1}{C_k} - \frac{1}{A_k}\right)\right]\dot{i}. \tag{2.28}$$

The boundary condition (2.7) can thus finally be written in the form

$$\frac{\partial w_k}{\partial x} - i\frac{\partial w_k}{\partial y} = -\frac{i}{2}\int_0^s J_k(s)\,ds - iC_{3k}. \tag{2.29}$$

The flexural and torsional rib rigidities A_k and C_k entering in (2.28) depend on the shape, dimensions, and material of the rib. The flexural rigidity A_k is

$$A_k = E_k I_{nk}, \tag{2.30}$$

where E_k is the modulus of elasticity of the rib material, I_{nk} is the moment of inertia of the rib cross section relative to its principal axis lying in the xy-plane and coinciding with the normal n to γ_k.

The torsional rigidity is

$$C_k = 2\mu_k \iint_{\Omega_k} \Phi(x, y)\,d\Omega = \mu_k I_{\tau k}, \tag{2.31}$$

where μ_k is the modulus of shear of the rib material, Ω_k is the cross-sectional area of the rib, $\Phi(x, y)$ is Prandtl's stress function (cf. Novozhilov /89/, p. 249), determined when solving Saint-Venant's problem of the torsion of a prismatic bar.

The ratio between the torsional and flexural rigidities of the rib is

$$\frac{C_k}{A_k} = \frac{\mu_k I_{\tau k}}{E_k I_{nk}} = \frac{I_{\tau k}}{2(1 + \nu_k) I_{nk}}, \tag{2.32}$$

where ν_k is Poisson's ratio for the rib material.

§ 3. ADDITIONAL NOTES. DERIVATION OF SOME FORMULAS

The integration constants C_{4k} in (2.22) and V_{bk}^0 in (2.23) will be determined from the condition that the strains and deflections of the ring be single-valued. Since the magnitudes $\theta_{\tau k}$ and θ_{nk} must regain their initial values after a complete pass along contour γ_k, we find from (2.18) that

$$\int_{\gamma_k} \left(\frac{L_{\tau k}}{C_k} - i \frac{L_{nk}}{A_k} \right) \dot{t} ds = 0. \tag{2.33}$$

Since by (2.26) and (2.28),

$$\int_{\gamma_k} \left(\frac{L_{\tau k}}{C_k} - i \frac{L_{nk}}{A_k} \right) \dot{t} ds = \frac{1}{2} \int_{\gamma_k} J_k(s)\, ds, \tag{2.34}$$

we obtain

$$\int_{\gamma_k} J_k(s)\, ds = 0. \tag{2.35}$$

This follows also from (2.29) as a consequence of the single-valuedness of the plate strains.

The deflection of the k-th rib is

$$\Delta_k(s) = -\int_0^s \theta_{nk} ds + C_{5k}, \tag{2.36}$$

where $C_{5k} = \Delta_k(0)$ is a real constant.

Substitution for θ_{nk} in (2.36) from (2.27) yields

$$\Delta_k(s) = -\operatorname{Re} \int_0^s \frac{i\dot{t}}{2} \left[\int_0^s J_k(s)\, ds + 2C_{3k} \right] ds + C_{5k}. \tag{2.37}$$

The condition that the deflections of the rib be single-valued is therefore

$$\operatorname{Im} \int_{\gamma_k} \left[\int_0^s J_k(s)\, ds \right] dt = 0. \tag{2.38}$$

Integrating the last equation by parts, and taking (2.35) into account, we obtain

$$\operatorname{Im} \int_{\gamma_k} t J_k(s)\, ds = 0. \tag{2.39}$$

The constants C_{4k} and V_{bk}^0 are determined from (2.35) and (2.39) whose solution yields

$$\left.\begin{aligned} V_{bk}^0 &= \frac{1}{v_k}\,\mathrm{Im}\int_{\gamma_k} u_k^*(s)\,I_k(s)\,ds \\ C_{4k} &= -iV_{bk}^0\bar{t}_{0k} + \int_{\gamma_k} u_{k2}(s)\,I_k(s)\,ds + \int_{\gamma_k} v_{k2}(s)\,\overline{I_k(s)}\,ds \end{aligned}\right\}, \tag{2.40}$$

where

$$\left.\begin{aligned} u_k^*(s) &= \left(\frac{1}{C_k}+\frac{1}{A_k}\right)[(a_k a_k' + \bar{b}_k\bar{a}_k') - t(a_k^2 - b_k\bar{b}_k)] + \\ &+ \left(\frac{1}{C_k}-\frac{1}{A_k}\right)[\bar{t}(a_k^2 - b_k\bar{b}_k) - (a_k\bar{a}_k' + b_k a_k')]\,\dot{t}^2 \\ u_{k2}(s) &= \frac{1}{a_k^2 - b_k\bar{b}_k}\left\{(b_k\bar{c}_k - a_k c_k)\frac{u_k^*(s)}{v_k} + \right. \\ &\left. + a_k\left(\frac{1}{C_k}+\frac{1}{A_k}\right) - b_k\left(\frac{1}{C_k}-\frac{1}{A_k}\right)\dot{t}^2\right\} \\ v_{k2}(s) &= \frac{1}{a_k^2 - b_k\bar{b}_k}\left\{(a_k c_k - b_k\bar{c}_k)\frac{\overline{u_k^*(s)}}{v_k} - \right. \\ &\left. - a_k\left(\frac{1}{C_k}-\frac{1}{A_k}\right)\dot{\bar{t}}^2 + b_k\left(\frac{1}{C_k}+\frac{1}{A_k}\right)\right\} \\ v_k &= a_k'(b_k\bar{c}_k - a_k c_k) + \bar{a}_k'(\bar{b}_k c_k - a_k\bar{c}_k) + c_k'(a_k^2 - b_k\bar{b}_k) \end{aligned}\right\} \tag{2.41}$$

$$\left.\begin{aligned} a_k &= l_k\left(\frac{1}{C_k}+\frac{1}{A_k}\right), \quad b_k = \left(\frac{1}{C_k}-\frac{1}{A_k}\right)\int_{\gamma_k}\dot{\bar{t}}^2 ds \\ c_k &= \left(\frac{1}{C_k}-\frac{1}{A_k}\right)\int_{\gamma_k} t\dot{\bar{t}}^2 ds - l_k\bar{z}_k^0\left(\frac{1}{C_k}-\frac{1}{A_k}\right) \\ a_k' &= \left(\frac{1}{C_k}+\frac{1}{A_k}\right)\bar{z}_k^0 l_k - \left(\frac{1}{C_k}-\frac{1}{A_k}\right)\int_{\gamma_k}\bar{t}\,\dot{t}^2 ds \\ c_k' &= \left(\frac{1}{C_k}-\frac{1}{A_k}\right)\int_{\gamma_k}(t^2\dot{\bar{t}}^2 + \bar{t}^2\dot{t}^2)ds - 2\left(\frac{1}{C_k}+\frac{1}{A_k}\right)\int_{\gamma_k} t\bar{t}\,ds \end{aligned}\right\}, \tag{2.41a}$$

l_k is the length of the rib γ_k, z_k^0 is the coordinate of its center of gravity. The function $I_k(s)$ is expressed through $m_k(s)$ and $p_k(s)$ by (2.25).

The internal forces and moments (bending and twisting moments and shearing force) in the rib γ_k are determined by means of (2.25) from the function $I_k^*(s)$, through which the moments $L_{\tau k}$, L_{nk}, and the shearing force V_{bk} acting on the rib are expressed, respectively, by (2.26) and (2.19):

$$L_{\tau k} = \mathrm{Re}\,I_k^*(s), \quad L_{nk} = -\,\mathrm{Im}I_k^*(s),$$

$$V_{bk} = \frac{1}{\varrho_k}\mathrm{Re}\,I_k^*(s) - \mathrm{Im}I_k^{*\prime}(s).$$

The internal forces and moments in the rib can also be expressed by other formulas, which are sometimes more convenient.

We derive these formulas by comparing (2.18) with (2.7):

$$\int_0^s \left(\frac{L_{\tau k}}{C_k} - i\frac{L_{nk}}{A_k}\right)\dot{\bar{t}}\,ds + C_{3k} = i\left(\frac{\partial w_k}{\partial x} - i\frac{\partial w_k}{\partial y}\right) = 2i\frac{\partial w_k}{\partial \bar{t}}. \tag{2.42}$$

Differentiation of this equation yields

$$\left(\frac{L_{\tau k}}{C_k} - i\frac{L_{nk}}{A_k}\right)d\bar{t} = 2id\left(\frac{\partial w_k}{\partial \bar{t}}\right) = 2i\left[\frac{\partial}{\partial t}\left(\frac{\partial w_k}{\partial \bar{t}}\right)dt + \frac{\partial}{\partial \bar{t}}\left(\frac{\partial w_k}{\partial \bar{t}}\right)d\bar{t}\right].$$

Hence,

$$\frac{L_{nk}}{A_k} + i\frac{L_{\tau k}}{C_k} = -2\left[\frac{dt}{d\bar{t}}\cdot\frac{\partial^2 w_k}{\partial t^2} + \frac{\partial^2 w_k}{\partial t\,\partial \bar{t}}\right]. \tag{2.43}$$

Since $\frac{dt}{d\bar{t}} = \dot{t}^2$, and also $\frac{\partial^2}{\partial t\,\partial \bar{t}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{1}{4}\Delta$, we obtain from (2.43) a new formula, which expresses the moments in the rib through the deflections w_k:

$$\frac{L_{nk}}{A_k} + i\frac{L_{\tau k}}{C_k} = -\frac{1}{2}\left(\Delta w_k + 4\dot{t}^2\frac{\partial^2 w_k}{\partial t^2}\right). \tag{2.44}$$

In view of (2.4) and (2.5), we can replace w_k by w_0 in (2.42) and (2.44).

It is easily seen from the derivation of (2.44) that this equation follows from (2.7), which states that the strains in the plate and in the stiffening rib are equal. It can also be proved that if (2.44) is satisfied, then (2.7) will also be satisfied.

Hence, (2.44) is equivalent to the condition that the strains in the plate and in the stiffening rib be equal.

Equation (2.44) can also be represented in a different form. The following relationships follow from (1.7):

$$\left.\begin{aligned} \Delta w &= -\frac{1}{D(1+\nu)}(M_x + M_y) = -\frac{1}{D(1+\nu)}(M_\tau + M_n) \\ \dot{t}^2\frac{\partial^2 w}{\partial t^2} &= \frac{e^{-2i\alpha}}{4}\left[\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2} + 2i\frac{\partial^2 w}{\partial x\,\partial y}\right] = \\ &= \frac{e^{-2i\alpha}}{4D(1-\nu)}[M_x - M_y - 2iH_{xy}] = \\ &= \frac{1}{4D(1-\nu)}[M_n - M_\tau + 2iH_{\tau n}] \end{aligned}\right\}, \tag{2.45}$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ is the plate rigidity, α is the angle between the normal n to the contour γ_k and the x-axis, measured counterclockwise from this axis; M_x, M_y, M_n, M_τ are bending moments, and H_{xy}, $H_{\tau n}$ twisting moments in S_0 (or S_k) at points of the contour γ_k.

Substituting (2.45) in (2.44), and taking into account that $H_{\tau n} = -H_{n\tau}$, we obtain

$$\frac{L_{nk}}{A_k} + i\frac{L_{\tau k}}{C_k} = \frac{1}{D(1-\nu^2)}[M_\tau - \nu M_n + i(1+\nu)H_{n\tau}]. \tag{2.46}$$

Separating the real and imaginary parts in (2.46), we find two more formulas expressing the moments in the rib through the moments in the plate adjacent to it:

$$\left.\begin{aligned} L_{nk} &= \frac{A_k}{D(1-\nu^2)}(M_\tau - \nu M_n) \\ L_{\tau k} &= \frac{C_k}{D(1-\nu)}H_{n\tau} \end{aligned}\right\}. \tag{2.47}$$

The shearing force in any section of the rib is by (2.19):

$$V_{bk} = \frac{1}{D(1-\nu^2)}\left[A_k\frac{\partial}{\partial s}(M_\tau - \nu M_n) + \frac{C_k(1+\nu)}{\varrho_k}H_{n\tau}\right]. \tag{2.48}$$

The quantities L_{nk} and $L_{\tau k}$ (2.47) have definite physical meanings, and their magnitudes are therefore independent of the selection of the domain S_0 or S_k in which the moments M_n, M_τ, and $H_{n\tau}$ are considered. It follows then from (2.47) that the right-hand side of (2.46) is also independent of the domain in which the moments acting on Γ are considered, and is thus invariant with respect to transition from domain S_k into domain S_0 across γ_k. Hence,

$$\begin{aligned} &\left\{\frac{1}{Eh^3}[M_\tau - \nu M_n + i(1+\nu)H_{n\tau}]\right\}^- = \\ &= \left\{\frac{1}{Eh^3}[M_\tau - \nu M_n + i(1+\nu)H_{n\tau}]\right\}^+, \end{aligned} \tag{2.49}$$

where the plus and minus signs correspond to the boundary values when $z \to t$, respectively, from the left and right of γ_k, i. e., in S_0 or S_k.

The derivation of (2.49) shows that this formula is valid also when the domains S_0 and S_k are respectively formed by plates of different rigidities D_0 and D_k.

For a plate of constant thickness it follows from (2.49) that

$$H_{n\tau}^+ = H_{n\tau}^-, \qquad (M_\tau - \nu M_n)^+ = (M_\tau - \nu M_n)^-.$$

The last equality can also be written in the form

$$\frac{M_\tau^+ - M_\tau^-}{M_n^+ - M_n^-} = \nu.$$

This means that the ratio of the jumps in the moments M_τ and M_n during transition across γ_k is equal to Poisson's ratio for the plate. This formula can be used to check the accuracy of approximate solutions.

Equation (2.49) is independent of the rigidity of the rib, and therefore remains valid even when no rib exists (absolutely flexible rib), i. e., for stepped plates. This formula was derived independently for the latter case by Sheremet'ev /182/ using a different procedure.

In the particular case of a circular stiffening rib, of radius R, reinforcing a circular or annular plate loaded axisymmetrically, we have

$$\left.\begin{aligned} L_{\tau k} &= H_{n\tau}^{+} = H_{n\tau}^{-} = 0 \\ L_{nk} &= R(M_n^{+} - M_n^{-}) \end{aligned}\right\}. \tag{2.50}$$

It follows therefore from (2.49) that for $(Eh^3)^{+} = (Eh^3)^{-}$

$$L_{nk} = \frac{R}{\nu}(M_\tau^{+} - M_\tau^{-}).$$

Using (2.50), we find from (2.46) for $M_n^{-} = 0$ in the axisymmetrical case, i. e., when an unloaded ring reinforces the edge of a circular hole in the plate,

$$\left(\frac{M_n}{M_\tau}\right)^{+} = \frac{\delta_1}{1 - \nu^2 + \delta_1 \nu}. \tag{2.51}$$

Here, $\delta_1 = \frac{A}{RD}$ is the relative flexural rigidity of the ring. It follows from (2.51) that $\left(\frac{M_n}{M_\tau}\right)^{+}$ increases monotonically from zero to $\frac{1}{\nu}$ when δ_1 increases from zero to infinity, and that $M_n^{+} = M_\tau^{+}$ for $\delta_1 = 1 + \nu$.

If $M_n^{+} = 0$, i. e., the rib (free of external forces and moments) reinforces the edge of a circular plate, we obtain from (2.46)

$$\left(\frac{M_\tau}{M_n}\right)^{-} = \frac{\nu^2 + \delta_1 \nu - 1}{\delta_1}. \tag{2.52}$$

Hence, $(M_\tau/M_n)^{-}$ increases monotonically from $-\infty$ to ν when δ_1 increases from zero to infinity; when $\delta_1 = 1 - \nu$, $M_\tau^{-} = M_n^{-}$, and when $\delta_1 = \frac{1 - \nu^2}{\nu}$, $M_\tau^{-} = 0$. When $\delta_1 = 0$ (no rib), $M_n^{-} = 0$, as ought to be expected.

Equations (2.51) and (2.52) are valid for any axisymmetrical load.

§ 4. CONDITIONS OF EQUILIBRIUM OF THE STIFFENING RIB

The rib γ_k is loaded by the moments $m_k(s)$ and forces $p_k(s)$ given by (2.3). Since the rib is in equilibrium, the corresponding principal vector and principal moment must vanish:

$$\int_{\gamma_k} p_k(s)\, ds = 0, \tag{2.53}$$

$$\int_{\gamma_k} m_k(s)\dot{\bar{t}}\,ds - i\int_{\gamma_k} t p_k(s)\,ds = 0. \tag{2.54}$$

Integrating by parts the second term in (2.54) and using (2.53), we obtain

$$\int_{\gamma_k} t p_k(s)\,ds = -\int_{\gamma_k} \dot{t}\left[\int_0^{s_1} p_k(s_2)\,ds_2\right] ds_1. \tag{2.55}$$

Substituting (2.55) in (2.54) and introducing the conjugated magnitudes, we obtain

$$\int_{\gamma_k}\left[m_k(s_1) - i\int_0^{s_1} p_k(s_2)\,ds_2\right]\dot{\bar{t}}\,ds_1 = 0. \tag{2.56}$$

Comparison with (2.25) yields

$$\int_{\gamma_k} dI_k(s) = 0$$

or

$$I_k(l_k) = I_k(0) = 0. \tag{2.57}$$

Thus, the requirement that the function $I_k(t)$ be single-valued follows from the equilibrium condition for the rib.

§ 5. THEOREM ON THE POTENTIAL ENERGY AND UNIQUENESS THEOREM

In order to derive the variational equations of Lagrange and Castigliano, which may be useful in approximately solving, by variational methods, problems of bending of plates with stiffening ribs, we have to prove the potential-energy theorem. Consider the integral, extending over the boundary $L = \sum_{j=1}^{m+1} L_j$ of domain S of a plate (Figure 6) carrying a tranverse load $q(x, y)$:

$$U = \int_L \left[w\left(N_n + \frac{\partial H_{n\tau}}{\partial s}\right) - \frac{\partial w}{\partial n} M_n\right] ds \tag{2.58}$$

(n is the external normal to domain S_0).

Here M_n, $H_{n\tau}$, and N_n are, respectively, the bending and twisting moments and shearing forces, defined by (1.56).

We find from (2.3):

$$0 \equiv \sum_{k=1}^{l}\int_{\gamma_k}\left\{w\left[p_k(s) + p_{kk}(s) - p_{0k}(s)\right] + \frac{\partial w}{\partial n}\left[m_k(s) + m_{kk}(s) - m_{0k}(s)\right]\right\} ds, \tag{2.59}$$

where

$$\left.\begin{aligned} m_{kk} = M_{n_1}^-,\ p_{kk} = N_{n_1}^- + \frac{\partial H_{n_1\tau}^-}{\partial s} \text{ on } \gamma_k \text{ in } S_k \\ m_{0k} = M_n,\ p_{0k} = N_{n_1} + \frac{\partial H_{n_1\tau}}{\partial s_1} = -\left(N_n + \frac{\partial H_{n\tau}}{\partial s}\right) \text{on } \gamma_k \text{in} S_0 \end{aligned}\right\}. \tag{2.60}$$

Here m_k and p_k are moments and forces acting on the stiffening rib γ_k; n_1 is the external normal to domain S_k; $ds_1 = -ds$. Adding (2.59) to (2.58), we obtain

$$U = \int_L \left[w\left(N_n + \frac{\partial H_{n\tau}}{\partial s}\right) - \frac{\partial w}{\partial n} M_n \right] ds + \sum_{k=1}^{l} \int_{\gamma_k} \left[w\left(N_n + \frac{\partial H_{n\tau}}{\partial s}\right) - \right.$$

$$\left. - \frac{\partial w}{\partial n} M_n \right] ds + \sum_{k=1}^{l} \int_{\gamma_k} \left[w\left(N_{n_1}^- + \frac{\partial H_{n_1\tau}^-}{\partial s}\right) - \frac{\partial w}{\partial n_1} M_{n_1}^- \right] ds +$$

$$+ \sum_{k=1}^{l} \int_{\gamma_k} \left[w p_k(s) + \frac{\partial w}{\partial n} m_k(s) \right] ds. \tag{2.61}$$

We introduce the symbols

$$U_1 = \int_{L_0} \left[w\left(N_n + \frac{\partial H_{n\tau}}{\partial s}\right) - \frac{\partial w}{\partial n} M_n \right] ds, \tag{2.62}$$

$$U_2 = \sum_{k=1}^{l} \int_{\gamma_k} \left[w\left(N_{n_1}^- + \frac{\partial H_{n_1\tau}^-}{\partial s}\right) - \frac{\partial w}{\partial n_1} M_{n_1}^- \right] ds, \tag{2.63}$$

$$U_3 = \sum_{k=1}^{l} \int_{\gamma_k} \left[w p_k(s) + \frac{\partial w}{\partial n} m_k(s) \right] ds, \tag{2.64}$$

where $L_0 = L + \sum_{k=1}^{l} \gamma_k$ is the boundary of domain S_0.

We can then write (2.61) as follows:

$$U = U_1 + U_2 + U_3. \tag{2.65}$$

Let us evaluate U_1, integrating by parts:

$$\int_{L_0} \frac{\partial H_{n\tau}}{\partial s} w ds = H_{n\tau} w \Big|_{L_0} - \int_{L_0} H_{n\tau} \frac{\partial w}{\partial s} ds. \tag{2.66}$$

The first term on the right-hand side of (2.66) represents the increment of the product $wH_{n\tau}$ during a complete pass around L_0, and vanishes since $H_{n\tau}$

and w are single-valued. Hence

$$\int_{L_0} \frac{\partial H_{n\tau}}{\partial s} w ds = -\int_{L_0} H_{n\tau} \frac{\partial w}{\partial s} ds. \tag{2.67}$$

We can therefore write (2.62) as follows:

$$U_1 = \int_{L_0} \left[N_n w - M_n \frac{\partial w}{\partial n} - H_{n\tau} \frac{\partial w}{\partial s} \right] ds. \tag{2.68}$$

Substituting (1.56) in (2.68), we obtain

$$\begin{aligned} U_1 = \int_{L_0} \Big\{ &[N_x \cos(n, x) + N_y \cos(n, y)] w - [M_x \cos(n, x) + \\ &+ H_{xy} \cos(n, y)] \left[\frac{\partial w}{\partial n} \cos(n, x) + \frac{\partial w}{\partial s} \cos(n, y) \right] - [M_y \cos(n, y) + \\ &+ H_{xy} \cos(n, x)] \left[\frac{\partial w}{\partial n} \cos(n, y) - \frac{\partial w}{\partial s} \cos(n, x) \right] \Big\} ds. \end{aligned} \tag{2.69}$$

Separating real and imaginary parts in (1.74) [after multiplying by $e^{-i\alpha}$], we obtain

$$\left.\begin{aligned} \frac{\partial w}{\partial n} \cos(n, x) + \frac{\partial w}{\partial s} \cos(n, y) &= \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial n} \cos(n, y) - \frac{\partial w}{\partial s} \cos(n, x) &= \frac{\partial w}{\partial y} \end{aligned}\right\}. \tag{2.70}$$

Hence, (2.69) becomes

$$\begin{aligned} U_1 = -\int_{L_0} \Big[&\left(M_x \frac{\partial w}{\partial x} + H_{xy} \frac{\partial w}{\partial y} - N_x w \right) \cos(n, x) + \left(M_y \frac{\partial w}{\partial y} + \right. \\ &\left. + H_{xy} \frac{\partial w}{\partial x} - N_y w \right) \cos(n, y) \Big] ds. \end{aligned} \tag{2.71}$$

With the aid of Green's formula we can transform the line integral (2.71) into an area integral over S_0:

$$\begin{aligned} U_1 = -\int\int_{S_0} \Big[&\left(\frac{\partial M_x}{\partial x} + \frac{\partial H_{xy}}{\partial y} - N_x \right) \frac{\partial w}{\partial x} + \left(\frac{\partial M_y}{\partial y} + \frac{\partial H_{xy}}{\partial x} - N_y \right) \frac{\partial w}{\partial y} - \\ &- \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} \right) w + \left(M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2} + 2 H_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \Big] dx dy. \end{aligned} \tag{2.72}$$

The equilibrium equations for a thin plate element are (cf. for instance /61/, § 130):

$$\left.\begin{aligned} \frac{\partial M_x}{\partial x}+\frac{\partial H_{xy}}{\partial y}-N_x&=0\\ \frac{\partial M_y}{\partial y}+\frac{\partial H_{xy}}{\partial x}-N_y&=0\\ \frac{\partial N_x}{\partial x}+\frac{\partial N_y}{\partial y}&=-q(x,y)\end{aligned}\right\} \tag{2.73}$$

The bending and twisting moments are, respectively, (cf. (1.7))

$$\left.\begin{aligned} M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right),\quad M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\\ H_{xy}=-D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\end{aligned}\right\}. \tag{2.74}$$

Hence, (2.72) becomes

$$U_1=-\iint_{S_0} q(x,\ y)\,w\,dx\,dy+2\iint_{S_0} W\,dx\,dy, \tag{2.75}$$

where

$$2W=D\left[\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)^2+(1-\nu^2)\left(\frac{\partial^2 w}{\partial y^2}\right)^2+2(1-\nu)\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right].$$

The function W represents the unit potential energy of bending strain of a thin plate, obtained when the shearing forces are ignored. Generally W is larger than zero, vanishing only when

$$\frac{\partial^2 w}{\partial x^2}=\frac{\partial^2 w}{\partial y^2}=\frac{\partial^2 w}{\partial x\partial y}=0, \tag{2.76}$$

i. e., when the deflection w is a linear function of the coordinates. In other words, $W=0$ only if w represents the rigid-body displacement of the plate. It follows then from (2.76) that the strain components (1.3) vanish, i. e.,

$$\varepsilon_x=\varepsilon_y=\gamma_{xy}=0. \tag{2.77}$$

Evaluating U_2 in the same way, we obtain

$$U_2=-\sum_{k=1}^{l}\left\{\iint_{S_k} q(x,\ y)\,w\,dx\,dy-2\iint_{S_k} W\,dx\,dy\right\}.$$

Addition of this to (2.75) yields

$$U_1+U_2=-\iint_{S} q(x,\ y)\,w\,dx\,dy+2\iint_{S} W\,dx\,dy. \tag{2.78}$$

We shall now evaluate U_3. It follows from (2.11) and (2.19) that

$$p_k(s) = -\frac{dV_{bk}}{ds} = -\frac{d}{ds}\left(\frac{L_{\tau k}}{\varrho_k} + \frac{dL_{nk}}{ds}\right), \tag{2.79}$$

$$m_k(s) = \frac{L_{nk}}{\varrho_k} - \frac{dL_{\tau k}}{ds}. \tag{2.80}$$

Substitution of (2.79) and (2.80) in (2.64) yields

$$U_3 = \sum_{k=1}^{l}\left\{\int_{\gamma_k}\frac{\partial w}{\partial n}\left(\frac{L_{nk}}{\varrho_k} - \frac{dL_{\tau k}}{ds}\right)ds - \int_{\gamma_k} w d\left(\frac{L_{\tau k}}{\varrho_k} + \frac{\partial L_{nk}}{ds}\right)\right\}.$$

Integrating the subtrahend by parts, and remembering that $w\left(\frac{L_{\tau k}}{\varrho_k} + \frac{dL_{nk}}{ds}\right)$ is single-valued, we obtain

$$U_3 = \sum_{k=1}^{l}\int_{\gamma_k}\left[\frac{\partial w}{\partial n}\left(\frac{L_{nk}}{\varrho_k} - \frac{\partial L_{\tau k}}{\partial s}\right) + \frac{\partial w}{\partial s}\left(\frac{L_{\tau k}}{\varrho_k} + \frac{dL_{nk}}{ds}\right)\right]ds. \tag{2.81}$$

Using (2.6), we can rewrite (2.81) in the form

$$U_3 = \sum_{k=1}^{l}\left\{\int_{\gamma_k}\left(\frac{L_{nk}}{\varrho_k}\theta_{\tau k} - \frac{L_{\tau k}}{\varrho_k}\theta_{nk}\right)ds - \int_{\gamma_k}(\theta_{\tau k}dL_{\tau k} + \theta_{nk}dL_{nk})\right\}. \tag{2.82}$$

Integrating the subtrahend by parts, noting that $\theta_{\tau k}$, θ_{nk}, $L_{\tau k}$, and L_{nk} are single-valued, we obtain

$$U_3 = \sum_{k=1}^{l}\int_{\gamma_k}\left[\left(\frac{d\theta_{\tau k}}{ds} - \frac{\theta_{nk}}{\varrho_k}\right)L_{\tau k} + \left(\frac{d\theta_{nk}}{ds} + \frac{\theta_{\tau k}}{\varrho_k}\right)L_{nk}\right]ds. \tag{2.83}$$

The relationships of Clebsch (2.14) and Kirchhoff (2.10) enable us to write U_3 finally in the form

$$U_3 = \sum_{k=1}^{l}\int_{\gamma_k}\left(\frac{L_{\tau k}^2}{C_k} + \frac{L_{nk}^2}{A_k}\right)ds. \tag{2.84}$$

The integrand in (2.84) represents the doubled potential strain energy per unit length of a thin bar when the shearing forces are ignored. Obviously, $U_3 \geqslant 0$, vanishing only when $L_{\tau k} = L_{nk} = 0$.

Substituting (2.78) and (2.84) in (2.65), we find

$$U = -\iint_S q(x, y)\, w dx dy + 2\iint_S W dx dy + \sum_{k=1}^{l}\int_{\gamma_k}\left(\frac{L_{\tau k}^2}{C_k} + \frac{L_{nk}^2}{A_k}\right)ds. \tag{2.85}$$

This expresses mathematically the theorem on the potential strain energy of a body in equilibrium under the action of a given load. Using (2.58), (2.85) can be rewritten as follows:

$$V = \frac{1}{2}\iint_S q(x, y)\, w\, dx dy + \frac{1}{2}\int_L \left[w\left(N_n + \frac{\partial H_{n\tau}}{\partial s}\right) - \frac{\partial w}{\partial n} M_n \right] ds, \qquad (2.86)$$

where

$$V = \sum_{k=1}^{l} \frac{1}{2} \int_{\gamma_k} \left(\frac{L_{\tau k}^2}{C_k} + \frac{L_{nk}^2}{A_k} \right) ds + \iint_S \frac{D}{2} \left[\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)^2 + \right.$$

$$\left. + (1-\nu^2) \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 2(1-\nu) \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] dx dy \qquad (2.87)$$

is the potential strain energy of a plate with stiffening ribs.

Thus, in the case of a thin plate, reinforced by stiffening ribs whose behavior is described by the theory of small deformations of thin curved bars, the theorem on the potential energy can be expressed mathematically as follows by (2.86):

Neglecting the transverse forces and taking into account only the energy of the bending and twisting moments, the potential strain energy of a plate with stiffening ribs, which is in equilibrium under the action of a given load, is equal to half the work done by the external forces when passing from the unstressed state to the state of equilibrium.

We shall now prove the uniqueness theorem. Assume first that two solutions are possible for the problem of the equilibrium of a plate with stiffening ribs under given conditions on the boundary L. Let w', L_{nk}, $L_{\tau k}$ be the plate deflections and the internal moments of the rib γ_k $(k = 1, 2, \ldots, l)$ corresponding to the first solution, and w'', $L_{nk}^{''}$, $L_{\tau k}^{''}$ the corresponding magnitudes for the second solution. The differences between these quantities are:

$$w = w'' - w',$$

$$L_{nk} = L_{nk}^{''} - L_{nk}^{'}, \quad L_{\tau k} = L_{\tau k}^{''} - L_{\tau k}^{'}.$$

The function w satisfies obviously the Sophie Germain equation for $q(x, y) = 0$, i. e., when no transverse load acts on the plate.

Setting $q(x, y) = 0$ in (2.85) and (2.58), we thus obtain for the difference between the two solutions w, L_{nk}, and $L_{\tau k}$

$$2\iint_S W dx dy + \sum_{k=1}^{l} \int_{\gamma_k} \left(\frac{L_{\tau k}^2}{C_k} + \frac{L_{nk}^2}{A_k} \right) ds = \int_L \left[w\left(N_n + \frac{\partial H_{n\tau}}{\partial s}\right) - \frac{\partial w}{\partial n} M_n \right] ds. \qquad (2.88)$$

We shall consider now the following basic problems.

1) First basic problem. The external bending moments and shearing forces per unit contour length are specified on L.

2) Second basic problem. The deflections of the plate edge and the slope of the bent middle surface (the derivative of the deflection with respect to the normal) are specified on L.

3) Third basic problem. The edge deflections and the external bending moments per unit contour length are specified on L.

4) Mixed problem. The conditions of the first basic problem are specified on one part of L (L'), those of the second basic problem on another part of L (L''), and those of the third basic problem on the remaining part of L (L''').

Since both our solutions satisfy the boundary conditions on L, their differences must satisfy zero conditions on the boundary, i.e.,

for the first basic problem

$$M_n = N_n + \frac{\partial H_{n\tau}}{\partial s} = 0 \text{ on } L;$$

for the second basic problem

$$w = \frac{\partial w}{\partial n} = 0 \text{ on } L;$$

for the third basic problem

$$w = M_n = 0 \text{ on } L;$$

for the mixed problem

$$\left.\begin{aligned} M_n = N_n + \frac{\partial H_{n\tau}}{\partial s} = 0 \text{ on } L' \\ w = \frac{\partial w}{\partial n} = 0 \text{ on } L'' \\ w = M_n = 0 \text{ on } L''' \end{aligned}\right\}. \tag{2.89}$$

Thus, in all these four cases the right-hand side of (2.88) vanishes, and therefore

$$2\iint_S W\,dx\,dy + \sum_{k=1}^{l}\int_{\gamma_k}\left(\frac{L_{\tau k}^2}{C_k} + \frac{L_{nk}^2}{A_k}\right)ds = 0.$$

The last condition is, however, fulfilled only when

$$W = 0 \text{ everywhere in } S, \tag{2.90}$$

$$L_{\tau k} = L_{nk} = 0 \text{ everywhere on } \gamma_k (k = 1, 2, \ldots, l). \tag{2.91}$$

We see from (2.77) that condition (2.90) is equivalent to the vanishing of the strain components of the plate for the difference of the two solutions; hence the corresponding moments and shearing forces acting on the plate must also be zero. Condition (2.91) means, according to (2.19), that on the rib γ_k the internal moments and the shearing forces V_{bk}, corresponding

to the differences of the two solutions, vanish. Hence, the strains $\delta\omega_{nk}$ and $\delta\omega_{\tau k}$ in (2.10) must also be zero. Thus, the two solutions are identical since they give identical moments, shearing forces, and strains in the plate and in the rib $\gamma_k (k = 1, 2, \ldots, l)$. The uniqueness theorem is thus proved. The proof given here is similar to the proof of Kirchhoff's Uniqueness Theorem for the basic problem of the theory of elasticity.

The identity of the stresses (strains) of the two solutions of our problem does not exclude differences in the deflections. Condition (2.90) means that the differences in the deflections, corresponding to the two solutions, represents only the rigid-body displacement of the plate and is, according to (2.76), a linear function of the coordinates, i.e.,

$$w = ax + by + c. \tag{2.92}$$

By (2.18), condition (2.91) is equivalent to

$$\theta_{\tau k} - i\theta_{nk} = C_{3k}\dot{t}.$$

Hence

$$\theta_{nk} = \mathrm{Re}(iC_{3k}\dot{t}).$$

It follows that the deflection Δ_k of rib γ_k, corresponding to the difference of the two solutions, is

$$\Delta_k = -\int \theta_{nk}ds + c' = -\mathrm{Re}\int iC_{3k}dt + c' = \frac{1}{2}[i(\bar{C}_{3k} - C_{3k})x + (C_{3k} + \bar{C}_{3k})y] + c'. \tag{2.93}$$

This means that for rib γ_k the difference between the two solutions represents a rigid-body displacement. According to (2.4), $w|_{\gamma_k} = \Delta_k$, and therefore

$$\frac{1}{2}i(\bar{C}_{3k} - C_{3k}) = a, \quad \frac{1}{2}\left(C_{3k} + \bar{C}_{3k}\right) = b, \quad c' = c. \tag{2.94}$$

Rigid-body displacement of the entire structure (plate with stiffening ribs) has obviously no effect on the stresses and strains in the plate and ribs, and will henceforth be ignored.

For the second, third, and mixed basic problems even this difference disappears since the deflections are specified on the entire boundary L or on part of it.

The following conclusion follows from the uniqueness theorem.

If the transverse load $q(x, y)$ vanishes and in addition: either a) the external bending moments and shearing forces are zero; or b) the deflections and their normal derivatives at all points of the plate edge are zero; or c) the deflections at the edge and the external bending moments are zero; or d) conditions similar to (2.89) are fulfilled, then the bending moments, shearing forces, and strains will vanish in the entire plate and in the rib γ_k.

The uniqueness theorem is proved similarly when the stiffening ribs are located at the plate edge.

§ 6. LAGRANGE'S VARIATIONAL EQUATION FOR A PLATE WITH STIFFENING RIBS

The potential energy of bending of a plate with stiffening ribs is given by (2.87). If we assume, for simplicity, that the plate is subjected only to a transverse load $q(x, y)$, then Lagrange's variational equation becomes

$$-\delta V + \iint_S q\delta w dx dy = 0. \tag{2.95}$$

where δw is a virtual deflection of the plate, i. e., a deflection compatible with the geometrical constraints imposed on the plate; δV is the variation of the function V defined by (2.87).

The deflection w is subjected to the geometrical conditions

$$w = 0 \text{ on } L, \tag{2.96}$$

if the plate is simply supported along its edge;

$$\frac{\partial w}{\partial n} = 0 \text{ on } L, \tag{2.97}$$

$$w = 0 \text{ on } L, \tag{2.98}$$

if the plate is built-in along its edge;

$$w^+ = w^-, \quad \frac{\partial w^+}{\partial n} = \frac{\partial w^-}{\partial n} \text{ on } \gamma_k, \tag{2.99}$$

$$w = \Delta_k, \quad \frac{\partial w}{\partial n} = \theta_{\tau k} \quad \text{on } \gamma_k, \tag{2.100}$$

where the plus and minus signs indicate that the corresponding magnitude belongs to domain S_0 and S_k, respectively.

Conditions (2.96) to (2.100) impose certain restrictions on the variation δw, namely:

if the plate is hinged along L,

$$\delta w = 0 \text{ on } L; \tag{2.101}$$

if the plate is built-in along L,

$$\delta w = 0 \text{ and } \delta\left(\frac{\partial w}{\partial n}\right) = 0 \text{ on } L. \tag{2.102}$$

In addition,

$$\delta w^- = \delta w^+, \quad \frac{\partial}{\partial n}(\delta w^-) = \frac{\partial}{\partial n}(\delta w^+) \text{ on } \gamma_k \tag{2.103}$$

$$(k = 1, 2, \ldots, l).$$

Using Kirchhoff's formula (/62/, p. 283), and taking (2.103) and (2.3) into account, we obtain from (2.87)

$$\delta V = D\iint (\Delta\Delta w)\,\delta w\,dx\,dy - \int_L M_n \frac{\partial}{\partial n}(\delta w)\,ds + \int_L \left(N_n + \right.$$

$$\left. + \frac{\partial H_{n\tau}}{\partial s}\right)\delta w\,ds - \int_\Gamma m_k(s)\frac{\partial}{\partial n}(\delta w)\,ds - \int_\Gamma p_k(s)\,\delta w\,ds + \sum_{k=1}^{i}\delta V_k. \qquad (2.104)$$

where

$$\delta V_k = \delta\left\{\frac{1}{2}\int_{V_k}\left(\frac{L_{\tau k}^2}{C_k} + \frac{L_{nk}^2}{A_k}\right)ds\right\}.$$

To determine δV_k we first transform the expression for V_k. With the aid of the formulas of Kirchhoff (2.10) and Clebsch (2.14) we obtain

$$V_k = \frac{1}{2}\int_{V_k}\left[C_k(\delta\omega_{\tau k})^2 + A_k(\delta\omega_{nk})^2\right]ds = \frac{1}{2}\int_{V_k}\left\{C_k\left[\frac{d}{ds}\left(\frac{\partial w}{\partial n}\right) + \right.\right.$$

$$\left.\left. + \frac{1}{\varrho_k}\cdot\frac{\partial w}{\partial s}\right]^2 + A_k\left[\frac{1}{\varrho_k}\cdot\frac{\partial w}{\partial n} - \frac{d}{ds}\left(\frac{\partial w}{\partial s}\right)\right]^2\right\}ds. \qquad (2.105)$$

Thus

$$\delta V_k = \int_{V_k}\left\{C_k\left[\frac{d}{ds}\left(\frac{\partial w}{\partial n}\right) + \frac{1}{\varrho_k}\cdot\frac{\partial w}{\partial s}\right]\left[\frac{d}{ds}\cdot\frac{\partial}{\partial n}(\delta w) + \frac{1}{\varrho_k}\cdot\frac{\partial}{\partial s}(\delta w)\right] + \right.$$

$$\left. + A_k\left[\frac{1}{\varrho_k}\cdot\frac{\partial w}{\partial n} - \frac{d}{ds}\left(\frac{\partial w}{\partial s}\right)\right]\left[\frac{1}{\varrho_k}\cdot\frac{\partial}{\partial n}(\delta w) - \frac{d}{ds}\cdot\frac{\partial}{\partial s}(\delta w)\right]\right\}ds.$$

This becomes by virtue of (2.14) and (2.10):

$$\delta V_k = \int_{V_k}\left\{L_{\tau k}\left[\frac{d}{ds}\cdot\frac{\partial}{\partial n}(\delta w) + \frac{1}{\varrho_k}\cdot\frac{\partial}{\partial s}(\delta w)\right] + L_{nk}\left[\frac{1}{\varrho_k}\cdot\frac{\partial}{\partial n}(\delta w) - \frac{d}{ds}\cdot\frac{\partial}{\partial s}(\delta w)\right]\right\}ds. \qquad (2.106)$$

Integrating by parts, we obtain for a closed contour (since the functions are single-valued)

$$\left.\begin{aligned}
\int_{V_k} L_{\tau k}\,d\frac{\partial}{\partial n}(\delta w) &= -\int_{V_k}\frac{\partial}{\partial n}(\delta w)\frac{dL_{\tau k}}{ds}\,ds \\
\int_{V_k} L_{nk}\,d\frac{\partial}{\partial s}(\delta w) &= -\int_{V_k}\frac{d}{ds}(\delta w)\cdot\frac{dL_{nk}}{ds}\,ds = \int_{V_k}\delta w\,d\left(\frac{dL_{nk}}{ds}\right) \\
\int_{V_k}\frac{L_{\tau k}}{\varrho_k}\cdot\frac{\partial}{\partial s}(\delta w)\,ds &= -\int_{V_k}\delta w\cdot\frac{d}{ds}\left(\frac{L_{\tau k}}{\varrho_k}\right)ds
\end{aligned}\right\}. \qquad (2.107)$$

Substituting (2.107) in (2.106), we obtain

$$\delta V_k = \int_{\gamma_k} \left\{ \left(\frac{L_{nk}}{\varrho_k} - \frac{dL_{\tau k}}{ds} \right) \frac{\partial}{\partial n} \delta w - \delta w \frac{d}{ds} \left(\frac{L_{\tau k}}{\varrho_k} + \frac{dL_{nk}}{ds} \right) \right\} ds. \tag{2.108}$$

Substitution of (2.108) in (2.104), and of the result in (2.95), yields

$$\iint_S (D\Delta\Delta w - q)\,\delta w dx dy - \int_L M_n \frac{\partial(\delta w)}{\partial n} ds + \int_L \left(N_n + \frac{\partial H_{n\tau}}{\partial s} \right) \delta w ds +$$
$$+ \int_\Gamma \left(\frac{L_{nk}}{\varrho_k} - \frac{dL_{\tau k}}{ds} - m_k \right) \frac{\partial(\delta w)}{\partial n} ds - \int_\Gamma \frac{d}{ds} \left(\frac{L_{\tau k}}{\varrho_k} + \frac{dL_{nk}}{ds} - V_{bk} \right) \delta w ds = 0, \tag{2.109}$$

where

$$V_{bk} = -\int_0^s p_k(s)\,ds + V_{bk}^0. \tag{2.110}$$

Integrating by parts the last integral in (2.109), we obtain the following variational equation for the bending of a thin plate reinforced by stiffening ribs:

$$\iint_S (D\Delta\Delta w - q)\,\delta w dx dy - \int_L M_n \frac{\partial(\delta w)}{\partial n} ds + \int_L \left(N_n + \frac{\partial H_{n\tau}}{\partial s} \right) \delta w ds +$$
$$+ \int_\Gamma \left(\frac{L_{nk}}{\varrho_k} - \frac{dL_{\tau k}}{ds} - m_k \right) \frac{\partial(\delta w)}{\partial n} ds + \int_\Gamma \left(\frac{L_{\tau k}}{\varrho_k} + \frac{dL_{nk}}{ds} - V_{bk} \right) \frac{\partial(\delta w)}{\partial s} ds = 0. \tag{2.111}$$

Since the variations δw inside the plate are arbitrary, we obtain from (2.111):

1) the Sophie Germain equation for transverse bending of a thin plate

$$D\Delta\Delta w = q, \tag{2.112}$$

2) the equilibrium equation (2.19) for an element of the stiffening rib belonging to Γ

$$\left. \begin{aligned} \frac{dL_{\tau k}}{ds} - \frac{L_{nk}}{\varrho_k} + m_k(s) = 0 \\ \frac{dL_{nk}}{ds} + \frac{L_{\tau k}}{\varrho_k} - V_{bk} = 0 \end{aligned} \right\}, \tag{2.113}$$

3) the boundary condition on L

$$-\int_L M_n \frac{\partial(\delta w)}{\partial n} ds + \int_L \left(N_n + \frac{\partial H_{n\tau}}{\partial s} \right) \delta w ds = 0. \tag{2.114}$$

If the edge L is built-in the variation δw on it must satisfy (2.102); hence, the boundary condition (2.114) must be satisfied by proper selection of the

function $w(x, y)$, and (2.111) reduces to the following variational equation:

$$\iint_S (D\Delta\Delta w - q)\,\delta w\,dx\,dy + \int_\Gamma \left(\frac{L_{nk}}{\varrho_k} - \frac{dL_{\tau k}}{ds} - m_k\right)\frac{\partial(\delta w)}{\partial n}\,ds + \\ + \int_\Gamma \left(\frac{L_{\tau k}}{\varrho_k} + \frac{dL_{nk}}{ds} - V_{bk}\right)\frac{\partial(\delta w)}{\partial s}\,ds = 0. \tag{2.115}$$

If the edge L is hinged, the variation δw must satisfy (2.101); the following condition then follows from (2.114):

$$M_n = 0 \quad \text{on } L. \tag{2.116}$$

If the plate edge L is free, the variation δw is arbitrary, and we obtain from (2.114) the boundary conditions

$$M_n = 0,\ N_n + \frac{\partial H_{n\tau}}{\partial s} = 0. \tag{2.117}$$

When solving approximately the problem by proceeding from Lagrange's variational equation (2.95), it is convenient to express V by substituting for V_k from (2.105):

$$V = \iint_S \frac{D}{2}\left[(\Delta w)^2 - 2(1-\nu)\left\{\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right\}\right]dx\,dy + \\ + \sum_{k=1}^{t}\frac{1}{2}\int_{\gamma_k}\left\{C_k\left[\frac{d}{ds}\left(\frac{\partial w}{\partial n}\right) + \frac{1}{\varrho_k}\cdot\frac{\partial w}{\partial s}\right]^2 + A_k\left[\frac{1}{\varrho_k}\cdot\frac{\partial w}{\partial n} - \frac{d}{ds}\left(\frac{\partial w}{\partial s}\right)\right]^2\right\}ds. \tag{2.118}$$

When (2.95) is used for solving problems of the bending of plates with stiffening ribs by the Ritz method, it is not necessary to satisfy beforehand the statical boundary conditions (2.113), (2.116), and (2.117), since they are satisfied automatically. The accuracy increases with the number of terms taken in the expression for the deflection

$$w = \sum_m c_m w_m(x, y) \tag{2.119}$$

and with the appropriateness of the functions $w_m(x, y)$ selected. The geometrical conditions (2.96) to (2.98) must be satisfied beforehand. In addition, the conditions (2.99) inside the plate must be satisfied beforehand by a suitable selection of $w_m(x, y)$.

§ 7. APPLICATION OF CASTIGLIANO'S VARIATIONAL EQUATION TO THE PROBLEM OF BENDING OF A PLATE WITH STIFFENING RIBS

The potential energy of a plate with stiffening ribs is given by (2.87), which can be written as follows: (/ 62/, p. 294):

$$V = \frac{1}{2(1-\nu^2)D}\iint_S [M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)H_{xy}^2]\,dxdy + \\ + \sum_{k=1}^{t}\frac{1}{2}\int_{\gamma_k}\left(\frac{L_{\tau k}^2}{C_k} + \frac{L_{nk}^2}{A_k}\right)ds. \tag{2.120}$$

The moments M_x, M_y, H_{xy}, $L_{\tau k}$, L_{nk} are interrelated by (2.19) and (2.73):

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 H_{xy}}{\partial x \partial y} + q = 0. \tag{2.121}$$

Let us, in addition to the actual state of stress in the ribbed plate, consider another statically possible state

$$M_x + \delta M_x,\ M_y + \delta M_y,\ H_{xy} + \delta H_{xy},$$
$$L_{\tau k} + \delta L_{\tau k},\quad L_{nk} + \delta L_{nk}, \tag{2.122}$$

which also satisfies the equilibrium conditions

$$\frac{\partial^2}{\partial x^2}(M_x + \delta M_x) + \frac{\partial^2}{\partial y^2}(M_y + \delta M_y) + 2\frac{\partial^2}{\partial x \partial y}(H_{xy} + \delta H_{xy}) + q = 0, \tag{2.123}$$

$$\left.\begin{aligned} \frac{L_{nk} + \delta L_{nk}}{\varrho_k} - \frac{d}{ds}(L_{\tau k} + \delta L_{\tau k}) - [m_k(s) + \delta m_k(s)] = 0 \\ \frac{L_{\tau k} + \delta L_{\tau k}}{\varrho_k} + \frac{d}{ds}(L_{nk} + \delta L_{nk}) - (V_{bk} + \delta V_{bk}) = 0 \end{aligned}\right\}. \tag{2.124}$$

From (2.121) and (2.123) it follows that

$$\frac{\partial^2}{\partial x^2}(\delta M_x) + \frac{\partial^2}{\partial y^2}(\delta M_y) + 2\frac{\partial^2}{\partial x \partial y}(\delta H_{xy}) = 0, \tag{2.125}$$

and from (2.19) and (2.124) that

$$\left.\begin{aligned} \frac{\delta L_{nk}}{\varrho_k} - \frac{d}{ds}(\delta L_{\tau k}) - \delta m_k(s) = 0 \\ \frac{\delta L_{\tau k}}{\varrho_k} + \frac{d}{ds}(\delta L_{nk}) - \delta V_{bk} = 0 \end{aligned}\right\}, \tag{2.126}$$

where

$$\delta V_{bk} = -\int_0^S \delta p_k(s)\,ds + \delta V_{bk}^0. \tag{2.127}$$

The variation δV of the potential energy is by (2.120):

$$\delta V = \frac{1}{D(1-\nu^2)} \iint_S [(M_x - \nu M_y)\,\delta M_x + (M_y - \nu M_x)\,\delta M_y + 2(1+\nu)\,H_{xy}\delta H_{xy}]\,dx\,dy + \sum_{k=1}^{l} \delta V_k, \tag{2.128}$$

where

$$\delta V_k = \int_{\gamma_k} \left[\frac{L_{\tau k}}{C_k}\,\delta(L_{\tau k}) + \frac{L_{nk}}{A_k}\,\delta(L_{nk}) \right] ds. \tag{2.129}$$

Equation (2.128) can be written as follows (/62/, p. 296) [this is evident from a comparison of (2.62) with (2.75)]:

$$\delta V = \int_{L_0} \left[\delta\left(N_n^+ + \frac{\partial H_{n\tau}^+}{\partial s}\right) w^+ - \delta M_n^+ \frac{\partial w^+}{\partial n} \right] ds + \int_{\Gamma} \left[\delta M_n^- \frac{\partial w^-}{\partial n} + - \delta\left(N_n^- + \frac{\partial H_{n\tau}^-}{\partial s}\right) w^- \right] ds + \sum_{k=1}^{l} \delta V_k, \tag{2.130}$$

where

$$L_0 = L + \Gamma.$$

We shall now determine δV_k. Using (2.10) and (2.14), which are valid for the actual state of stress, (2.129) can be rewritten as follows:

$$\delta V_k = \int_{\gamma_k} [\delta\omega_{\tau k}\delta(L_{\tau k}) + \delta\omega_{nk}\delta(L_{nk})]\,ds = \int_{\gamma_k} \left[\left(\frac{d\theta_{\tau k}}{ds} - \frac{\theta_{nk}}{\varrho_k} \right) \delta L_{\tau k} + \left(\frac{d\theta_{nk}}{ds} + \frac{\theta_{\tau k}}{\varrho_k} \right) \delta L_{nk} \right] ds. \tag{2.131}$$

Integration by parts yields

$$\delta V_k = \int_{\gamma_k} \left[\theta_{\tau k} \left(\frac{1}{\varrho_k}\,\delta L_{nk} - \frac{d}{ds}\,\delta L_{\tau k} \right) - \theta_{nk} \left(\frac{1}{\varrho_k}\,\delta L_{\tau k} + \frac{d}{ds}\,\delta L_{nk} \right) \right] ds. \tag{2.132}$$

Since

$$\left.\begin{aligned} \theta_{\tau k} &= \frac{\partial w^+}{\partial n} = \frac{\partial w^-}{\partial n} = \frac{\partial w}{\partial n} \\ \theta_{nk} &= -\frac{\partial w^+}{\partial s} = -\frac{\partial w^-}{\partial s} = -\frac{\partial w}{\partial s} \end{aligned}\right\} \text{ on } \gamma_k, \tag{2.133}$$

we obtain, using (2.126)

$$\delta V_k = \int_{\gamma_k} \left[\frac{\partial w}{\partial n} \delta m_k(s) + \frac{\partial w}{\partial s} \delta V_{bk} \right] ds. \tag{2.134}$$

Integration by parts of the second term in (2.134) yields

$$\delta V_k = \int_{\gamma_k} \left[\frac{\partial w}{\partial n} \delta m_k(s) - w \delta p_k(s) \right] ds, \tag{2.135}$$

where

$$\left.\begin{aligned} \delta m_k(s) &= \delta M_n^+ - \delta M_n^- \\ -\delta p_k(s) &= \delta\left(N_n^- + \frac{\partial H_{n\tau}^-}{\partial s} \right) - \delta\left(N_n^+ + \frac{\partial H_{n\tau}^+}{\partial s} \right) \end{aligned}\right\}; \tag{2.136}$$

[cf. (2.3) and (2.60).]

Substitution of (2.135) in (2.130) finally yields

$$\delta V = \int_L \delta\left(N_n + \frac{\partial H_{n\tau}}{\partial s} \right) w ds - \int_L \delta M_n \frac{\partial w}{\partial n} ds. \tag{2.137}$$

If the plate is built-in along L,

$$w = 0, \quad \frac{\partial w}{\partial n} = 0 \qquad \text{on } L. \tag{2.138}$$

We obtain, therefore, from (2.137)

$$\delta V = 0. \tag{2.139}$$

If the plate is hinged along L, $\delta M_n = 0$ and $w = 0$ on L, and (2.139) is again valid. Finally, if the plate is free along L on which M_n and $\left(N_n + \frac{\partial H_{n\tau}}{\partial s} \right)$ are given, the variations of these magnitudes vanish on L, and (2.139) remains valid.

We have thus proved that Castigliano's equation takes the form (2.139) in the case of bending of a plate with ribs, δV being given by (2.137).

When we solve approximately the bending problem with the aid of (2.139), we actually define the shearing forces $N_x^+, N_y^+, N_x^-, N_y^-$, according to (2.73), by specifying $M_x^-, M_y^-, H_{xy}^-, M_x^+, M_y^+$ and H_{xy}^+.

Hence the loads $m_k(s)$ and $p_k(s)$ (2.3) on rib γ_k are also specified in this case, i.e., the internal moments $L_{\tau k}$ and L_{nk} in the rib, entering in the expression (2.129) for δV_k, are also given.

In fact, the latter are expressed through $m_k(s)$ and $p_k(s)$ by means of (2.26) and (2.25), the constants C_{4k} and V_{bk}^0 being determined from (2.40).

§ 8. REDUCTION TO THE BOUNDARY-VALUE PROBLEM OF THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

The solution of the problem formulated in § 1 reduces (cf. Chapter I) to determining the functions $\varphi_k(z)$ and $\psi_k(z)$ $(k = 0, 1, 2, \ldots, l)$ of the complex variable $z = x + iy$, regular, respectively, in each of the domains S_k $(k = 0, 1, 2, \ldots, l)$, except at the points where concentrated forces and moments are applied. These functions are determined from the boundary conditions (2.1), (2.3), (2.29), and (2.8) (cf. note on p. 62 at the end of this section). The deflection w of the plate in $S_k (k = 0, 1, 2, \ldots, l)$ is expressed through them by (1.23):

$$w_k = w_k^1 + w_k^0 = 2\,\mathrm{Re}\,[\bar{z}\varphi_k(z) + \chi_k(z)] + w_k^0(x, y), \tag{2.140}$$

where $\chi_k(z) = \int \psi_k(z)\,dz$; and $w_k^0(x, y)$ is a particular solution of the differential equation (1.13) of the bending of a thin plate:

$$\Delta\Delta w_k = \frac{1}{D}\, q_k(x, y). \tag{2.141}$$

In order to simplify the analysis without restricting its generality, we take as particular solution $w_k^0(x, y)$ $(k = 0, 1, 2, \ldots, l)$ that obtained when no stiffening ribs are provided. This solution (the basic deflection) will henceforth be assumed as known.

In this case

$$w_k^1 = 2\,\mathrm{Re}\,[\bar{z}\varphi_k(z) + \chi_k(z)] \tag{2.142}$$

represents the additional deflection of the plate, caused by the presence of the stiffening ribs. Certain additional strains and internal forces in the plate correspond to this additional deflection. The functions $\varphi_k(z)$ and $\psi_k(z)$ are obviously holomorphic in the corresponding domains $S_k (k = 0, 1, 2, \ldots, l)$.

We shall now express the boundary conditions (2.1), (2.3), (2.8), and (2.29) through the functions $\varphi_k(z)$ and $\psi_k(z)$. Condition (2.1) coincides with (1.55) which is equivalent to (1.65). Since, by hypothesis, $w_0^0(x, y)$ satisfies the boundary conditions (2.1) on L, i. e., $l^0(s) = l(s)$ on L, (1.65) reduces to

$$-\varkappa\,\overline{\varphi_0(\tau)} + \bar{\tau}\varphi_0'(\tau) + \psi_0(\tau) = -iC_{1j}\bar{\tau} + C_{2j} \text{ on } L_j \ (j = 1, 2, \ldots, m+1), \tag{2.143}$$

where τ is the complex coordinate of the point on L, $\varkappa = (3 + \nu)/(1 - \nu)$; C_{1j} and C_{2j} are, respectively, real and complex constants.

We rewrite (2.3) first in the form

$$-\int_0^s \left[m_k(s_1) - i\int_0^{s_1} p_k(s_2)\,ds_2\right]\frac{dx}{dt}\,ds_1 =$$

$$= -\int_0^s \left[m_{0k}(s_1) - i\int_0^{s_1} p_{0k}(s_2)\,ds_2\right]\frac{dx}{dt}\,ds_1 +$$

$$+\int_0^s \left[m_{kk}(s_1) - i\int_0^{s_1} p_{kk}(s_2)\,ds_2\right]\frac{dx}{dt}\,ds_1$$

or, using (2.25),

$$I_k(s) = I_{ok}(s) - I_{kk}(s).$$

The functions $I_{ok}(s)$ and $I_{kk}(s)$ are expressed through the analytic functions $\varphi_0(z)$, $\psi_0(z)$, $\varphi_k(z)$ and $\psi_k(z)$ by formulas similar to (1.65); the latter equation becomes

$$-\varkappa\overline{\varphi_0(t)} + \bar{t}\varphi_0(t) + \psi_0(t) = -\varkappa\overline{\varphi_k(t)} + \bar{t}\varphi_k'(t) + \psi_k(t) - \\ - \frac{I_k}{2D(1-\nu)} - \frac{1}{2D(1-\nu)}(iC_{6k}\bar{t} - C_{7k}) \text{ on } \gamma_k, \qquad (2.144)$$

where t is the complex coordinate of the point on contour Γ; C_{6k} and C_{7k} are, respectively, real and complex constants. In deriving (2.144) we made use of the continuity of the bending moments and shearing forces, corresponding to the basic deflection w^0, when crossing γ_k from S_0 to S_k (Figure 6). The functions $I_k = I_k(s) = I_k(t)$ are defined by (2.25). These functions depend on the jump in the forces and moments across γ_k, and are still unknown.

We shall now transform (2.8) by expressing its right- and left-hand sides similarly to (1.76). Since, by their physical meaning, the first derivatives, with respect to x and y, of the basic deflection w^0 are continuous when crossing γ_k from S_0 to S_k, we obtain

$$\overline{\varphi_0(t)} + \bar{t}\varphi_0'(t) + \psi_0(t) = \overline{\varphi_k(t)} + \bar{t}\varphi_k'(t) + \psi_k(t) \text{ on } \gamma_k. \qquad (2.145)$$

Finally, (2.29) is similarly reduced to

$$\overline{\varphi_k(t)} + \bar{t}\varphi_k'(t) + \psi_k(t) = \frac{1}{4i}\int_0^s J_k(s)\,ds + \frac{C_{3k}}{2i} - \frac{\partial w_k^0}{dt} \text{ on } \gamma_k, \qquad (2.146)$$

where

$$\frac{\partial w^0}{\partial t} = \frac{\partial w^0}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w^0}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{1}{2}\left(\frac{\partial w^0}{\partial x} - i\frac{\partial w^0}{\partial y}\right).$$

The boundary conditions (2.1), (2.3), (2.8), and (2.29) have thus been reduced to (2.143) through (2.146).

Condition (2.143) states that the additional forces in the plate, caused by the stiffening ribs, satisfy the zero boundary conditions on L, while (2.144) defines the jump in the additional internal forces in the plate across γ_k. Condition (2.145) states that the additional strains in adjacent plate domains along γ_k are equal, while (2.146) states that the strains of the ring and the plate are equal.

The right-hand side of (2.146) depends on the unknown strains of the ring, which are functions of $m_k(s)$ and $p_k(s)$ defined by (2.3). These are determined by integrating the basic equations of the theory of small deformations of thin curved bars.

The boundary conditions (2.143) to (2.146) are the initial conditions for solving the problem of the elastic equilibrium of a plate reinforced by thin curved stiffening ribs. We repeat that in these boundary conditions

$I_k (k=1, 2, \ldots, l)$ are still unknown functions. The unknown functions $\varphi_k(z)$ and $\psi_e(z)$ $(k = 0, 1, 2, \ldots, l)$ are defined and holomorphic, respectively, in each of the domains S_k into which S is divided by Γ.

These functions are discontinuous across Γ. We introduce two functions $\varphi(z)$ and $\psi(z)$ which are piecewise-holomorphic (/82/, Chapter VI) in S and discontinuous across Γ. The above boundary conditions then become

$$-\varkappa\overline{\varphi(\tau)} + \bar{\tau}\varphi'(\tau) + \psi(\tau) = -iC_{1j}\bar{\tau} + C_{2j} \text{ on } L_j \ (j=1, 2, \ldots, m+1), \tag{2.147}$$

$$-\varkappa\overline{\varphi^+(t)} + \bar{t}\varphi'^+(t) + \psi^+(t) = -\varkappa\overline{\varphi^-(t)} + \bar{t}\varphi'^-(t) + \psi^-(t) + (\varkappa+1)\overline{g(t)} \text{ on}\Gamma, \tag{2.148}$$

$$\overline{\varphi^+(t)} + \bar{t}\varphi'^+(t) + \psi^+(t) = \overline{\varphi^-(t)} + \bar{t}\varphi'^-(t) + \psi^-(t) \text{ on } \Gamma, \tag{2.149}$$

$$\overline{\varphi^-(t)} + \bar{t}\varphi'^-(t) + \psi^-(t) = P(t) - R(t) \text{ on } \Gamma. \tag{2.150}$$

The limits of the functions $\varphi(z)$, $\varphi'(z)$, and $\psi(z)$ when $z \to t$ from the left and from the right are denoted, respectively, by the superscripts "+" and "—". The functions $g(t)$, $R(t)$, and $P(t)$ assume the values

$$g_k(t) = -\frac{\bar{I}_k - iC_{6k}t - \bar{C}_{7k}}{8D}, \tag{2.151}$$

$$P_k(t) = \frac{1}{4i}\int_0^s J_k(s)\,ds, \tag{2.152}$$

$$\bar{R}_k(t) = \frac{\partial\omega_k^0}{\partial t}\Big|_{\gamma_k} - \frac{1}{2i}C_{3k} \tag{2.152a}$$

on each contour γ_k $(k = 1, 2, \ldots, l)$, respectively.

The unknown function $P_k(t)$ is easily expressed through $I_k(t)$ by means of (2.28) and (2.25), and vice versa. Thus,

$$P_k(t) = \frac{1}{4i}\int_0^s \left\{\left(\frac{1}{C_k}+\frac{1}{A_k}\right)[C_{4k} - iV_{bk}^0(\bar{t}-\bar{t}_0) + I_k(t)] + \right.$$
$$\left. + \left(\frac{1}{C_k}-\frac{1}{A_k}\right)\dot{\bar{t}}^2\,[\bar{C}_{4k} + iV_{bk}^0(t-t_0) + \overline{I_k(t)}]\right\} ds. \tag{2.153}$$

According to (2.57) and (2.151), the function $g(t)$ is single-valued on γ_k $(k = 1, 2, \ldots, l)$.

Using Sherman's method /193/, we consider first (2.148) and (2.149). Subtracting the first from the second, and using the conjugate quantities, we obtain

$$\varphi^+(t) - \varphi^-(t) = -g(t) \text{ on } \Gamma. \tag{2.154}$$

We then find from (2.149):

$$\psi^+(t) - \psi^-(t) = h(t) \text{ on } \Gamma. \tag{2.155}$$

where

$$h(t) = \overline{g(t)} + \bar{t}g'(t), \quad g'(t) = \frac{dg(t)}{dt} \tag{2.156}$$

The problem of determining $\varphi(z)$ and $\psi(z)$ has thus been reduced to solving (2.155) and (2.154). The solution is /193/

$$\left.\begin{aligned}\varphi(z) &= \varphi^0(z) - \frac{1}{2\pi i}\int_\Gamma \frac{g(t)\,dt}{t-z}\\ \psi(z) &= \psi^0(z) + \frac{1}{2\pi i}\int_\Gamma \frac{h(t)\,dt}{t-z}\end{aligned}\right\}, \tag{2.157}$$

where $\varphi^0(z)$ and $\psi^0(z)$ are unknown functions, holomorphic throughout S, which have to be determined.

Substituting (2.157) in (2.147), we obtain

$$-\varkappa\overline{\varphi^0(\tau)} + \bar{\tau}\varphi^{0\prime}(\tau) + \psi^0(\tau) = \Phi(\tau) - iC_1'\bar{\tau} + C_2', \text{ on } L_j \tag{2.158}$$
$$(j = 1, 2, \ldots, m+1),$$

where

$$\Phi(\tau) = \frac{\varkappa}{2\pi i}\int_\Gamma \frac{\overline{g(t)\,dt}}{t-\bar{\tau}} + \frac{\bar{\tau}}{2\pi i}\int_\Gamma \frac{g'(t)\,dt}{t-\tau} - \frac{1}{2\pi i}\int_\Gamma \frac{h(t)\,dt}{t-\tau}. \tag{2.159}$$

The functions $\varphi^0(z)$ and $\psi^0(z)$ are thus determined by solving the first basic problem of bending of a plate whose middle plane forms a multiply-connected domain S bounded by the contour L. After finding $\varphi^0(z)$ and $\psi^0(z)$ we obtain from (2.157) the piecewise-holomorphic functions $\varphi(z)$ and $\psi(z)$, expressed through the unknown function $g(t)$.

Substituting then $\varphi(z)$ and $\psi(z)$ in (2.150), whose right-hand side also contains $g(t)$, and using (2.152), (2.28), (2.25), and (2.151), we obtain on Γ an integral-differential equation for determining $g(t)$, through which $\varphi(z)$ and $\psi(z)$ are then found from (2.157).

Note. If the domain S is simply connected and the edge L is not free, but $w_0(s)$ and $\frac{\partial w_0}{\partial n}$ are specified on it according to (2.2), then, obviously, everything said remains true, except for the boundary condition (2.147), which is replaced by

$$\overline{\varphi(\tau)} + \bar{\tau}\varphi'(\tau) + \psi(\tau) = 0 \text{ on } L. \tag{2.160}$$

If the plate edge L is hinged, (2.147) is replaced by the appropriate boundary conditions /65/.

§ 9. DETERMINACY OF FUNCTIONS INTRODUCED

We shall now consider the determinacy of the piecewise-holomorphic functions $\varphi(z)$ and $\psi(z)$, the functions $\varphi^0(z)$ and $\psi^0(z)$ which are holomorphic

throughout S, and the function $g(t)$, introduced in the preceding section, when the state of stress in a plate with ribs or the deflections of this plate are given.

1) Let the state of stress be given. In this case (cf. § 2, Chapter I), the functions $\varphi_0(z)$, $\psi_0(z)$, $\chi_0(z)$, $\varphi_k(z)$, $\psi_k(z)$ and $\chi_k(z)$ $(k = 1, 2, \ldots, l)$ are respectively determined except for the expressions

$$ic_0 z + \alpha_0' + i\beta_0', \quad \alpha_0'' + i\beta_0'', \quad (\alpha_0'' + i\beta_0'')\, z + \alpha_0''' + i\beta_0''', \tag{2.161}$$

$$ic_k z + \alpha_k' + i\beta_k', \quad \alpha_k'' + i\beta_k'' \quad (\alpha_k'' + i\beta_k'')z + \alpha_k''' + i\beta_k''', \tag{2.162}$$

where $c_k, \alpha_k', \beta_k', \alpha_k'', \beta_k'', \alpha_k''', \beta_k'''$ $(k = 0, 1, 2, \ldots, l)$ are real constants. It is also obvious that, if we add to the above functions expressions (2.161) or (2.162), respectively, the stresses in the plate will not be affected: if we replace, respectively, the functions

$$\left.\begin{aligned} &\varphi_k(z) \text{ by } \varphi_k(z) + ic_k z + \alpha_k' + i\beta_k' \\ &\psi_k(z) \text{ by } \psi_k(z) + \alpha_k'' + i\beta_k'' \\ &\chi_k(z) \text{ by } \chi_k(z) + (\alpha_k'' + i\beta_k'')z + \alpha_k''' + i\beta_k''' \end{aligned}\right\}, \tag{2.163}$$

only the deflections

$$w_k = w_k^0 + 2\mathrm{Re}\,[\bar{z}\varphi_k(z) + \chi_k(z)], \tag{2.164}$$

will change, and we obtain

$$w_k + 2(\alpha_k' + \alpha_k'')x + 2(\beta_k' - \beta_k'')y + 2\alpha_k''' \tag{2.165}$$
$$(k = 0, 1, 2, \ldots, l)$$

instead of w_k.

The additional terms in (2.165) [cf. (1.30)] describe the rigid-body displacement of the plate in each domain S_k separately. However, the plate deflections must be continuous across γ_k from S_0 to S_k, i.e.,

$$\left.\begin{aligned} &w_0 = w_k \text{ on } \gamma_k \\ &(k = 1, 2, \ldots, l) \end{aligned}\right\}. \tag{2.166}$$

Substitution of (2.165) in (2.166) yields

$$(\alpha_k' + \alpha_k'')x + (\beta_k' - \beta_k'')y + \alpha_k''' = (\alpha_0' + \alpha_0'')x + (\beta_0' - \beta_0'')y + \alpha_0'''.$$

The following $3l$ relationships between the constants in (2.163) are then obtained:

$$\left.\begin{aligned} &\alpha_0' + \alpha_0'' = \alpha_k' + \alpha_k'' \\ &\beta_0' - \beta_0'' = \beta_k' - \beta_k'' \\ &\alpha_0''' = \alpha_k''' \end{aligned}\right\} \quad (k = 1, 2, \ldots, l). \tag{2.167}$$

Hence, of the $7(l+1)$ real constants appearing in (2.161) and (2.162), only $4l+7$ are independent, for instance

$$c_0,\ \alpha_0^*,\ \beta_0^*,\ \alpha_0^{*'},\ \beta_0^{*'},\ \alpha_0^{*''},\ \beta_0^{*''} \tag{2.168}$$

and

$$c_k,\ \beta_k^{*''},\ \alpha_k^{*'},\ \beta_k^{*'} \quad (k=1, 2, \ldots, l). \tag{2.169}$$

We shall now consider how substitution (2.163) affects the stresses and strains in rib γ_k. Since the additional terms in (2.165) represent only a rigid-body displacement of the plate, the corresponding additional moments and forces $m_k(s)$ and $p_k(s)$ vanish, so that the functions $I_k(s)$ given by (2.25) remain unchanged. The substitution leaves also unchanged the constants C_{4k}, V_{bk}^0, given by (2.40), and the functions $I_k^*(s)$ (2.25), $J_k(s)$ (2.28), L_{nk}, $L_{\tau k}$ (2.26), V_{bk} (2.23). In other words, substitution (2.163) does not affect the stresses in rib γ_k, as ought to be expected.

The additional deflections and slopes of the sections of rib γ_k, caused by the substitution (2.163), are determined from the condition that the deflections and strains of plate and rib be equal:

$$\Delta_k = w_k, \quad \dot{t}(\theta_{\tau k} - i\theta_{nk}) = i\frac{\partial w_k}{\partial x} + \frac{\partial w_k}{\partial y} \quad \text{on} \quad \gamma_k \tag{2.170}$$
$$(k=1, 2, \ldots, l).$$

Replacing w_k by (2.165) in (2.170), we obtain the following substitutions

$$\left.\begin{array}{l} \Delta_k \ \text{на}\ \Delta_k + 2(\alpha_k^* + \alpha_k^{*'})x + 2(\beta_k^* - \beta_k^{*'})y + 2\alpha_k^{*''}, \\ \dot{t}(\theta_{\tau k} - i\theta_{nk}) \ \text{на}\ \dot{t}(\theta_{\tau k} - i\theta_{nk}) + 2i(\alpha_k^* + \alpha_k^{*'}) + 2(\beta_k^* - \beta_k^{*'}) \end{array}\right\}. \tag{2.171}$$

The additional terms in (2.171) also describe only a rigid-body displacement of the rib γ_k.

Putting $t = t_{0k}$ in (2.18) (t_{0k} is the abscissa of point $s = O_k$, the point from which the arc lengths are measured on γ_k), we find that the substitution (2.163) causes the constant $C_{3k} = \dot{t}_{0k}(\theta_{\tau k}^0 - i\theta_{nk}^0)$ in (2.17) to be replaced by

$$\dot{t}_{0k}(\theta_{\tau k}^0 - i\theta_{nk}^0) + 2i(\alpha_k^* + \alpha_k^{*'}) + 2(\beta_k^* - \beta_k^{*'}). \tag{2.172}$$

Since the functions $\varphi_k(z)$, $\psi_k(z)$, and $\chi_k(z)$ are interrelated by the boundary conditions (2.143) to (2.146), it is of interest to determine how the substitution (2.163) affects their fulfilment.

Carrying out the substitution (2.163) in (2.145), we obtain, as ought to be expected, relationships coinciding with the first two equations (2.167). The same substitution in (2.146) leads to an identity if (2.172) is taken into account and it is remembered that $J_k(s)$, given by (2.28), is unchanged.

As a result of the substitution (2.163), the expression $[-\varkappa\overline{\varphi_k(z)} + \bar{z}\varphi_k'(z) + \psi_k(z)]$ is replaced by

$$-\varkappa\overline{\varphi_k(z)} + \bar{z}\varphi_k'(z) + \psi_k(z) + ic_k(\varkappa+1)\bar{z} + (\alpha_k^{*'} - \varkappa\alpha_k^{*}) + i(\beta_k^{*'} + \varkappa\beta_k^{*}) \tag{2.173}$$
$$(k=0, 1, 2, \ldots, l).$$

Thus, by a suitable selection of the constants c_k, $(\alpha_k'' - \varkappa\alpha_k')$, and $(\beta_k'' + \varkappa\beta_k')$ we can give any values to the constants C_{6k} and $C_{7k} = C_{7k}' + iC_{7k}''$, entering in (2.144), and also to the constant $C_{2,m+1} = C_{2,m+1}' + iC_{2,m+1}''$ in (2.143). The constant $C_{1,m+1}$ is determined during the solution of the problem from the condition that the deflections (2.142) be single-valued.

Thus, when the domain S is bounded, we can, by placing the origin of coordinates at any point in S_1, select the constants (2.168) and (2.169) so that, e. g., the following $(4l + 7)$ conditions be satisfied:

$$C_{2,m+1} = 0, \quad \varphi_0(z_0) = \chi_0(z_0) = \operatorname{Im}\varphi_0'(z_0) = 0, \tag{2.174}$$

$$C_{6k} = -V_{bk}^0, \quad C_{7k} = -C_{4k} - iV_{bk}^0 \bar{t}_{0k}, \quad \operatorname{Im}[\chi_k(z_k)] = 0 \,(k = 1, 2, \ldots, l), \tag{2.175}$$

where z_k is any point in S_k $(k = 0, 1, 2, \ldots, l)$. Instead of the condition $\varphi_0(z_0) = 0$ in (2.174) we can require that $\psi_0(z_0) = 0$. If the domain is simply-connected, the condition $\operatorname{Im}\varphi_0'(z_0) = 0$ can be replaced by the condition $C_{1,m+1} = 0$.

Conditions (2.174) and (2.175) define completely the functions $\varphi_k(z)$ and $\psi_k(z)$.

If the domain S is infinite, we can require that

$$\operatorname{Im}\varphi_0'(\infty) = 0, \quad C_{2,1} = 0, \quad \varphi_0(\infty) = 0, \quad \chi_0(\infty) = 0 \tag{2.176}$$

instead of (2.174) (assuming the stresses to be finite at infinity).

We can replace $\varphi_0(\infty) = 0$ in (2.176) by $\psi_0(\infty) = 0$. The conditions (2.175) and (2.174) or (2.176) eliminate completely the indeterminacy of the piecewise holomorphic functions $\Psi(z)$, $\chi(z)$. The initial freedom in determining the functions $g_k(t)$ is restricted by (2.175) which, inserted into (2.151), yields according to (2.25):

$$g_k(t) = -\frac{1}{8D}[\overline{l_k(s)} + iV_{bk}^0(t - t_0) + \bar{C}_{4k}] = -\frac{i}{8D}\overline{l_k^*(s)}, \tag{2.177}$$

where, as was shown above, the function $l_k^*(s)$ is completely determined.

Thus, when the stresses and the additional conditions (2.175) are given, the function $g(t)$, and therefore also $h(t)$ defined by (2.156), are completely defined. Since the piecewise-holomorphic functions $\varphi(z)$ and $\psi(z)$ are thus also completely defined, we conclude from (2.157) that all functions $\varphi^0(z)$ and $\psi^0(z)$, which are holomorphic in S, are exactly determined.

2) Consider now the case when the plate deflections are given. Since the state of stress of the plate is then completely defined, the indeterminacy of the functions considered is reduced. Obviously, there is no more general substitution than (2.163). We shall now determine what additional conditions are imposed on the constants by the substitution (2.163), if the deflections of the plate are given.

In this case the additional terms in (2.165) must vanish:

$$\alpha_k' + \alpha_k'' = 0, \quad \beta_k' - \beta_k'' = 0, \quad \alpha_k'' = 0 \,(k = 0, 1, 2, \ldots, l). \tag{2.178}$$

Condition (2.167) is in this case satisfied identically; hence, the constants (2.161) and (2.162) are linked by $3(l + 1)$ (and not $3l$) relationships,

so that only $4(l+1)$ of them are independent, for instance

$$\left.\begin{array}{l} c_0,\ \alpha_0^*,\ \beta_0^*,\ \beta_0^{**} \\ c_k,\ \beta_k^{**},\ \alpha_k^{*'},\ \beta_k^{*'}\ \ (k=1,2,\ldots,l) \end{array}\right\}, \tag{2.179}$$

which can be determined from the following conditions:

$$\varphi_0(z_0)=0,\quad \operatorname{Im}\chi_0(z_0)=0, \tag{2.180}$$

$$C_{6k}=-V_{bk}^0,\quad C_{7k}=-C_{4k}-iV_{bk}^0\,\bar{l}_{0k},\quad \operatorname{Im}[\chi_k(z_k)]=0 \tag{2.181}$$

$$(k=1,2,\ldots,l).$$

The above statements regarding the functions $g(t)$, $h(t)$, $\varphi_0(z)$ and $\psi_0(z)$ remain valid.

Note 1. Conditions (2.175) or (2.181) enable us to express through $g_k(t)$ the right-hand side of (2.150), i. e., the function $P_k(t)$ given by (2.153):

$$P_k(t)=2iD\int_0^s\left[\left(\frac{1}{C_k}+\frac{1}{A_k}\right)\overline{g_k(t)}+\left(\frac{1}{C_k}-\frac{1}{A_k}\right)\dot{t}^2 g_k(t)\right]ds. \tag{2.182}$$

Note 2. If

$$\frac{\partial w_k^0}{\partial t}=v_k=\text{const on }\gamma_k, \tag{2.183}$$

i. e., the contour $\Gamma=\sum_{k=1}^{l}\gamma_k$ of a plate without ribs undergoes only a rigid-body displacement, then obviously reinforcing the plate by an elastic rib along Γ does not affect its state of stress. In other words, stresses in the plate will not depend on the rigidity of the rib. In fact, under condition (2.183) all the boundary conditions (2.147) to (2.150) are satisfied if we assume that the additional stresses and strains in the plate, caused by the stiffening rib, vanish identically:

$$\varphi_0(z)=\varphi_k(z)=\alpha_0=\text{const},\quad \psi_0(z)=\psi_k(z)=\alpha_0'=\text{const},$$

$$C_{1,j}=0,\quad C_{2,j}=\alpha_0'-\varkappa\bar{\alpha}_0,\quad g_k(t)=0,\quad C_{3k}=2i(v_k+\bar{\alpha}_0+\alpha_0'), \tag{2.184}$$

$$(k=1,2,\ldots,l),\quad (j=1,2,\ldots,m+1).$$

We therefore conclude on the basis of the uniqueness theorem (cf. section 5) that (2.184) represents the solution of the problem under condition (2.183) which, in this case, is only a rigid-body displacement of the plate. The internal forces and moments in the stiffening rib vanish also (cf. (2.47) and (2.48)).

Note 3. If the basic deflections of the plate are such that on γ_k is satisfied the condition

$$\frac{d}{ds}\left(\frac{\partial w^0}{\partial t}\right) = 2iD\left[\left(\frac{1}{C_k}-\frac{1}{A_k}\right)\dot{t}^2 C_{\bullet 1} + \left(\frac{1}{C_k}+\frac{1}{A_k}\right)\overline{C}_{\bullet 1}\right]. \tag{2.185}$$

or

$$\frac{d}{ds}\left(\frac{\partial w^0}{\partial t}\right) = 4iD\,(a_k C_{\bullet 1}\dot{t}^2 + b_k \overline{C}_{\bullet 1}), \tag{2.186}$$

where a_k, b_k, $C_{\bullet 1}$ are constants, then a rib, whose rigidities are $A_k = \frac{1}{b_k - a_k}$ and $C_k = \frac{1}{b_k + a_k}$, does not affect the stresses in the plate.

In fact, under conditions (2.185) or (2.186) all the boundary conditions (2.147) to (2.150) are satisfied if we assume that

$$\left.\begin{gathered} \varphi_0(z) = 0, \quad \varphi_k(z) = \alpha_k = \text{const}, \quad C_{1,j} = 0 \\ \psi_0(z) = 0, \quad \psi_k(z) = -\overline{\alpha}_k = \text{const}, \quad C_{2,j} = 0 \\ g(t) = C_{\bullet 1} = \alpha \end{gathered}\right\} \tag{2.187}$$

(where α acquires the value α_k on γ_k).

We conclude on the basis of the uniqueness theorem that (2.187) is the solution of the problem under condition (2.185). This solution represents only a rigid-body displacement of the plate. We shall now show that in this case the load acting on the stiffening rib from the plate vanishes identically. In fact, we obtain from (2.177), by substituting $g(t) = C_{\bullet 1}$ and differentiating:

$$\frac{d}{ds} I_k(s) = iV^0_{bk}\dot{t}. \tag{2.188}$$

Substituting in (2.188) the value of $I_k(s)$ according to (2.25), we find

$$-m_k(s) + i\int_0^s p_k(s_1)\,ds_1 = iV^0_{bk}$$

Hence

$$m_k(s) \equiv 0, \quad p_k(s) \equiv 0,$$

which concludes the proof.

§10. CIRCULAR PLATE WITH CURVED STIFFENING RIBS. REDUCING THE PROBLEM TO FREDHOLM'S INTEGRAL EQUATION

Consider a circular plate of radius R_1 reinforced by l curved stiffening ribs. In this case (Figure 8), S is bounded and simply-connected, while S_0 is bounded and $(l+1)$-fold connected.

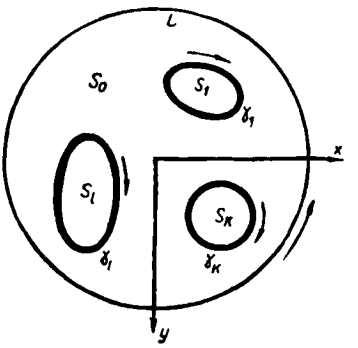


FIGURE 8.

It was shown in § 8 that this problem reduces to determining the piecewise holomorphic functions $\varphi(z)$ and $\psi(z)$ defined by (2.157). The functions $\varphi^0(z)$ and $\psi^0(z)$ entering in (2.157), holomorphic in the circle $r \leqslant R_1$, are determined by solving the first basic problem for the circle with the aid of (2.158), in which, because of the arbitrariness of the unknown functions, we can put $C_{1,l} = C_{2,l} = 0$. . Thus

$$-\varkappa\overline{\varphi^0(\tau)} + \bar{\tau}\varphi^{0\prime}(\tau) + \psi^0(\tau) = \Phi(\tau) \text{ on } L, \tag{2.189}$$

where the function $\Phi(\tau)$ is defined by (2.159).

Multiplying (2.189) by $\frac{1}{2\pi i}\cdot\frac{d\tau}{\tau - z}$ and integrating along the circle L, where $|z| < R_1$, we obtain

$$\psi^0(z) + \frac{R^2}{z}\varphi^{0\prime}(z) = \frac{1}{2\pi i}\int_L \frac{\Phi(\tau)d\tau}{\tau - z} + \frac{\alpha_1 R_1^2}{z} + \varkappa\bar{\alpha}_0, \tag{2.190}$$

where

$$\alpha_0 = \varphi^0(0), \quad \alpha_1 = \varphi^{0\prime}(0). \tag{2.191}$$

Using conjugate values in (2.189), we find

$$-\varkappa\varphi^0(\tau) + \tau\overline{\varphi^{0\prime}(\tau)} + \overline{\psi^0(\tau)} = \overline{\Phi(\tau)}. \tag{2.192}$$

Multiplying (2.192) by $\frac{1}{2\pi i}\cdot\frac{d\tau}{\tau - z}$ and integrating along L, where $|z| < R_1$, we obtain

$$\varkappa\varphi^0(z) = -\frac{1}{2\pi i}\int_L \frac{\overline{\Phi(\tau)}d\tau}{\tau - z} + \bar{\alpha}_0' + 2\bar{\alpha}_2 R_1^2 + \bar{\alpha}_1 z, \tag{2.193}$$

where

$$\alpha_0' = \psi^0(0), \quad 2\alpha_2 = \varphi^{0\prime\prime}(0). \tag{2.194}$$

We determine the integrals in the right-hand sides of (2.190) and (2.193) by introducing the following functions of z:

$$\left.\begin{aligned}\Phi_1(z)&=-\frac{\varkappa}{2\pi i}\int_\Gamma\frac{g(t)\,dt}{t-z},\quad \Phi_2(z)=-\frac{1}{2\pi i}\int_\Gamma\frac{h(t)\,dt}{t-z}\\ \Phi_3(z)&=\frac{R_1^2}{z}\cdot\frac{1}{2\pi i}\int_\Gamma\frac{g'(t)dt}{t-z}\end{aligned}\right\}. \tag{2.195}$$

In accordance with (2.151) and (2.57), we shall assume that the functions $g(t)$, $g'(t)$, and therefore also $h(t)$, are single-valued and continuous, and satisfy the Hölder condition (H).

The functions $\Phi_i(z)$ $(i=1,2,3)$, defined by (2.195), are holomorphic everywhere outside Γ, and therefore also everywhere outside L, and vanish at infinity.

We therefore have for $|z|<R$

$$\left.\begin{aligned}&\frac{1}{2\pi i}\int_L\frac{\Phi_k(\tau)\,d\tau}{\tau-z}=0\\ &\frac{1}{2\pi i}\int_L\frac{\overline{\Phi_k(\tau)}\,d\tau}{\tau-z}=\overline{\Phi}_k\left(\frac{R_1^2}{z}\right)=\overline{\Phi_k\left(\frac{R_1^2}{\bar z}\right)}\end{aligned}\right\}(k=1,2,3). \tag{2.196}$$

By (2.159) and (2.195)

$$\Phi(\tau)=\overline{\Phi_1(\tau)}+\Phi_2(\tau)+\Phi_3(\tau). \tag{2.197}$$

Substituting (2.197) in the integrals in the right-hand sides of (2.190) and (2.193), and using (2.196), we obtain

$$\left.\begin{aligned}&\frac{1}{2\pi i}\int_L\frac{\Phi(\tau)d\tau}{\tau-z}=\overline{\Phi_1\left(\frac{R_1^2}{\bar z}\right)}=\frac{\varkappa}{2\pi i}\int_\Gamma\frac{\overline{g(t)}\,d\bar t}{\bar t-\frac{R_1^2}{z}}\\ &\frac{1}{2\pi i}\int_L\overline{\Phi(\tau)}\frac{d\tau}{\tau-z}=\overline{\Phi_2\left(\frac{R_1^2}{\bar z}\right)}+\overline{\Phi_3\left(\frac{R_1^2}{\bar z}\right)}=\\ &=\frac{1}{2\pi i}\int_\Gamma\frac{\overline{h(t)}\,d\bar t}{\bar t-\frac{R_1^2}{z}}-\frac{z}{2\pi i}\int_\Gamma\frac{\overline{g'(t)}\,d\bar t}{\bar t-\frac{R_1^2}{z}}\end{aligned}\right\}. \tag{2.198}$$

We substitute (2.198) in (2.190) and (2.193). Then

$$\left.\begin{aligned}\varkappa\varphi^0(z)&=-\frac{1}{2\pi i}\int_\Gamma\frac{\overline{h(t)}-z\overline{g'(t)}}{\bar t-\frac{R_1^2}{z}}d\bar t+\bar\alpha_0'+2\bar\alpha_2R_1^2+\bar\alpha_1 z\\ \psi^0(z)+\frac{R_1^2}{z}\varphi^{0\prime}(z)&=\frac{\varkappa}{2\pi i}\int_\Gamma\frac{\overline{g(t)}\,d\bar t}{\bar t-\frac{R_1^2}{z}}+\frac{\alpha_1R_1^2}{z}+\varkappa\bar\alpha_0\end{aligned}\right\}. \tag{2.199}$$

The functions $\varphi^0(z)$ and $\psi^0(z)$ in (2.199) are determined by the function $g(t)$ except for some still unknown terms. The constants α_0, α_0', α_1, α_2 are determined from the condition that they are coefficients of powers of z, corresponding to the subscripts, in the expansions of $\varphi^0(z)$ and $\psi^0(z)$. We therefore set $z = 0$ in [(2.199) and] the equations obtained from it by differentiating with respect to z, and take into account (2.191) and (2.194). This yields:

$$\alpha_2 = 0, \quad \alpha_0' = \varkappa\bar{\alpha}_0. \tag{2.200}$$

$$\varkappa\alpha_1 - \bar{\alpha}_1 = \frac{1}{2\pi i R_1^2}\int_\Gamma \overline{h(t)}\,d\bar{t}. \tag{2.201}$$

Using (2.156), integrating by parts the right-hand side of the last equation, and noting that the function $g(t)$ is single-valued, we can reduce (2.201) to the form

$$\varkappa\alpha_1 - \bar{\alpha}_1 = \frac{1}{2\pi i R_1^2}\int_\Gamma [g(t)\,d\bar{t} - \overline{g(t)}\,dt].$$

Hence, α_1 is real:

$$\alpha_1 = \frac{1}{R_1^2(\varkappa - 1)}\cdot\frac{1}{2\pi i}\int_\Gamma [g(t)\,d\bar{t} - \overline{g(t)}\,dt]. \tag{2.202}$$

Substituting (2.200) and (2.202) in (2.199), we obtain

$$\left.\begin{aligned}
\varphi^0(z) &= \alpha_0 + \frac{z}{2\pi i\varkappa(\varkappa-1)R_1^2}\left[\int_\Gamma g(t)\,\frac{z\bar{t} - \varkappa R_1^2}{z\bar{t} - R_1^2}\,d\bar{t} + \right.\\
&\left. + \int_\Gamma \overline{g(t)}\,d\frac{\varkappa R_1^2(t-z) + z(R_1^2 - \bar{t}t)}{z\bar{t} - R_1^2}\right]\\
\psi^0(z) &= \alpha_0' - \frac{R_1^2}{z}\varphi^{0\prime}(z) + \frac{1}{2\pi i(\varkappa-1)z}\int_\Gamma g(t)\,d\bar{t} + \\
&+ \frac{1}{2\pi i(\varkappa-1)}\int_\Gamma \overline{g(t)}\left[\frac{\varkappa(\varkappa-1)\bar{t}^2}{\bar{t} - \frac{R_1^2}{z}} - \frac{1}{z}\right]dt
\end{aligned}\right\}. \tag{2.203}$$

We have thus expressed through $g(t)$ the functions $\varphi^0(z)$ and $\psi^0(z)$, entering in the expressions for the unknown piecewise-holomorphic functions $\varphi(z)$ and $\psi(z)$ defined by (2.157). The latter must satisfy the boundary condition (2.150), which have so far not been used. The right-hand part of (2.150) contains the function $P(t)$, and also the unknown constant C_{3k}. To eliminate this constant, we differentiate (2.150) with respect to s. This yields

$$\dot{\bar{t}}_0^2[\overline{\varphi'^-(t_0)} + \varphi'^-(t_0)] + \bar{t}_0\varphi''^-(t_0) + \psi'^-(t_0) = P'(t_0) - R'(t_0) \quad \text{on } \Gamma, \tag{2.204}$$

where $\Gamma = \sum_{k=1}^{l}\gamma_k$, t_0 is the complex coordinate of the point on Γ.

We use Sokhotskii's formula (/104/, p. 178) to derive the necessary boundary values of the functions (2.157) and their derivatives:

$$\left.\begin{aligned}
\varphi^-(t_0) &= \frac{1}{2}\,g(t_0) - \frac{1}{2\pi i}\int_\gamma \frac{g(t)\,dt}{t-t_0} + \varphi^0(t_0)\\
\psi^-(t_0) &= -\frac{1}{2}\,\overline{g(t_0)} - \frac{\bar t_0}{2}\,g'(t_0) + \frac{1}{2\pi i}\int_\gamma \frac{\overline{g(t)} + \bar t g'(t)}{t-t_0}\,dt + \\
&\quad + \psi^0(t_0)\\
\varphi^{*-}(t_0) &= \frac{1}{2}\,g'(t_0) - \frac{1}{2\pi i}\int_\gamma \frac{g'(t)\,dt}{t-t_0} + \varphi^{0*}(t_0)\\
\varphi^{**-}(t_0) &= \frac{1}{2}\,g''(t_0) - \frac{1}{2\pi i}\int_\gamma \frac{g''(t)\,dt}{t-t_0} + \varphi^{0**}(t_0)\\
\psi^{*-}(t_0) &= -\frac{1}{2}\,\overline{g'(t_0)}\,\dot{\bar t}_0^2 - \frac{1}{2}\,\dot t_0^2 g'(t_0) - \frac{1}{2}\,\bar t_0 g''(t_0) + \\
&\quad + \frac{1}{2\pi i}\int_\gamma [\dot t^2\,\overline{g'(t)} + \dot{\bar t}^2\,g'(t) + \bar t\,g''(t)]\,\frac{dt}{t-t_0} + \psi^{0*}(t_0)
\end{aligned}\right\}. \qquad (2.205)$$

The functions $\psi^0(t_0)$ and $\psi^{0*}(t_0)$ are obtained from (2.203) for $z = t_0$.

Substituting (2.205) in (2.204), we obtain:

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\gamma \left[1 + \frac{\dot t_0^2\,(t-t_0)}{\bar t - \bar t_0}\right]\dot{\bar t}^2\,\overline{g'(t)}\,\frac{dt}{t-t_0} + \\
&+ \frac{1}{2\pi i}\int_\gamma \left(\frac{\bar t - \bar t_0}{t-t_0} - \dot{\bar t}_0^2\right) g'(t)\,\frac{dt}{t-t_0} = 2iD\,\dot{\bar t}_0\left[\left(\frac{1}{C} + \frac{1}{A}\right)\overline{g(t_0)} + \right. \qquad (2.206)\\
&\left. + \left(\frac{1}{C} - \frac{1}{A}\right) g(t_0)\,\dot{\bar t}_0^2\right] - \Omega(t_0) - \left[\frac{d}{dt}\left(\frac{\partial w_k^0}{\partial \bar t}\right)\right]_{t=t_0},
\end{aligned}$$

where

$$\Omega(t_0) = \dot{\bar t}_0^2\,[\overline{\varphi^{0*}(t_0)} + \varphi^{0*}(t_0)] + \bar t_0 \varphi^{0**}(t_0) + \psi^{0*}(t_0). \qquad (2.207)$$

The rigidities A and C acquire on each contour γ_k, respectively, the values A_k and C_k $(k = 1, 2, \ldots, l)$.

We determine the function $\Omega(t_0)$ from (2.203).

Thus, for $z = t_0$,

$$\begin{aligned}
\varphi^0(t_0) = a_0 + \frac{t_0}{2\pi i \varkappa(\varkappa-1)R_1^2}\left[\int_\gamma g(t)\,\frac{t_0\bar t - \varkappa R_1^2}{t_0 \bar t - R_1^2}\,d\bar t + \right.\\
\left. + \int_\gamma \overline{g(t)}\,d\,\frac{\varkappa R_1^2(t-t_0) + t_0(R_1^2 - t\bar t)}{t_0 \bar t - R_1^2}\right].
\end{aligned}$$

Differentiating by t_0, we obtain

$$\varphi^{0\prime}(t_0) = \frac{1}{2\pi i \varkappa(\varkappa-1)R_1^2}\left\{\int_\gamma g(t)\,\frac{\varkappa R_1^4 + \bar{t}^2 t_0^2 - 2R_1^2 \bar{t} t_0}{(t_0\bar{t}-R_1^2)^2}\,d\bar{t} - \right.$$
$$\left. - \int_\gamma \overline{g(t)}\,d\left[t + \frac{(\varkappa-1)R_1^2(\bar{t}\,t_0^2 - 2R_1^2 t_0 + R_1^2 t)}{(t_0\bar{t}-R_1^2)^2}\right]\right\}. \tag{2.207a}$$

The right-hand side is integrated by parts, remembering that $g(t)$ is single-valued (cf. § 8):

$$\int_\gamma g(t)\,\frac{\varkappa R_1^4 + \bar{t}^2 t_0^2 - 2R_1^2 \bar{t} t_0}{(t_0\bar{t}-R_1^2)^2}\,d\bar{t} =$$
$$= \left[g(t)\int_\gamma \frac{\varkappa R_1^4 + \bar{t}^2 t_0^2 - 2R_1^2 \bar{t} t_0}{(t_0\bar{t}-R_1^2)^2}\,d\bar{t}\right]_\Gamma -$$
$$- \int_\gamma g'(t)\left[\int \frac{\varkappa R_1^4 + \bar{t}^2 t_0^2 - 2R_1^2 \bar{t} t_0}{(t_0\bar{t}-R_1^2)^2}\,d\bar{t}\right]dt =$$
$$= -\int_\gamma g'(t)\left[\bar{t} + \frac{(1-\varkappa)R_1^4}{(t_0\bar{t}-R_1^2)t_0}\right]dt,$$
$$\int_\gamma \overline{g(t)}\,d\left[t + \frac{(\varkappa-1)R_1^2(\bar{t}t_0^2 - 2R_1^2 t_0 + R_1^2 t)}{(t_0\bar{t}-R_1^2)^2}\right] =$$
$$= -\int_\gamma \left[\frac{(\varkappa-1)R_1^2(\bar{t}t_0^2 - 2R_1^2 t_0 + R_1^2 t)}{(t_0\bar{t}-R_1^2)^2} + t\right]\dot{t}^2\,\overline{g'(t)}\,dt.$$

Substituting these values in (2.207a), we obtain the function $\varphi^{0\prime}(t_0)$. The functions $\psi^{0\prime\prime}(t_0)$, $\psi^{0\prime}(t_0)$ are obtained similarly. Finally,

$$\left.\begin{aligned}
\varphi^{0\prime}(t_0) &= \frac{1}{\pi i}\int_\gamma g'(t)\,k_1(t,t_0)\,dt + \frac{1}{\pi i}\int_\gamma \overline{g'(t)}\,k_2(t,t_0)\,dt,\\
\varphi^{0\prime\prime}(t_0) &= \frac{1}{\pi i}\int_\gamma g'(t)\,k_3(t,t_0)\,dt + \frac{1}{\pi i}\int_\gamma \overline{g'(t)}\,k_4(t,t_0)\,dt,\\
\psi^{0\prime}(t_0) &= \frac{1}{\pi i}\int_\gamma g'(t)\,k_5(t,t_0)\,dt + \frac{1}{\pi i}\int_\gamma \overline{g'(t)}\,k_6(t,t_0)\,dt +\\
&\quad + \frac{R_1^2}{t_0^2}\varphi^{0\prime}(t_0) - \frac{R_1^2}{t_0}\varphi^{0\prime\prime}(t_0)
\end{aligned}\right| \tag{2.208}$$

where

$$\left.\begin{aligned}
k_1(t, t_0) &= -\frac{1}{2\varkappa(\varkappa-1)R_1^2}\left[\bar{t} - \frac{(\varkappa-1)R_1^4}{t_0(t_0\bar{t}-R_1^2)}\right] \\
k_2(t, t_0) &= -\frac{\dot{\bar{t}}^2(2R_1^2 t_0 - R_1^2 t - \bar{t}t_0^2)}{2\varkappa(\bar{t}t_0 - R_1^2)^2} + \frac{\dot{\bar{t}}^2}{2\varkappa(\varkappa-1)R_1^2} \\
k_3(t, t_0) &= \frac{R_1^2(R_1^2 - 2t_0\bar{t})}{2\varkappa t_0^2(t_0\bar{t}-R_1^2)^2} \\
k_4(t, t_0) &= \frac{\dot{\bar{t}}^3 R_1^2(R_1^2 - \bar{t}t)}{\varkappa(\bar{t}t_0 - R_1^2)^3} \\
k_5(t, t_0) &= \frac{\bar{t}}{2t_0^2(\varkappa-1)} \\
k_6(t, t_0) &= \frac{-\dot{\bar{t}}^2}{2(\varkappa-1)}\left[\frac{t}{t_0^2} + \frac{\varkappa(\varkappa-1)\bar{t}}{\bar{t}t_0 - R_1^2}\right]
\end{aligned}\right\}. \tag{2.209}$$

Substituting (2.208) in (2.207), we find

$$\Omega(t_0) = \frac{1}{\pi i}\int_\Gamma \Omega_1(t_0, t)\, g'(t)\, dt + \frac{1}{\pi i}\int_\Gamma \Omega_2(t_0, t)\, \overline{g'(t)}\, dt, \tag{2.210}$$

where

$$\left.\begin{aligned}
\Omega_1(t_0, t) &= \left(\dot{\bar{t}}_0^2 + \frac{R_1^2}{t_0^2}\right)k_1(t_0, t) - \dot{\bar{t}}_0^2\,\dot{\bar{t}}^2\,\overline{k_2(t_0 t)} + \\
&+ \left(\bar{t}_0 - \frac{R_1^2}{t_0}\right)k_2(t_0, t) + k_3(t_0, t) \\
\Omega_2(t, t) &= \left(\dot{\bar{t}}_0^2 + \frac{R_1^2}{t_0^2}\right)k_2(t_0, t) - \dot{\bar{t}}_0^2\,\dot{\bar{t}}^2\,\overline{k_1(t_0, t)} + \\
&+ \left(\bar{t}_0 - \frac{R_1^2}{t_0}\right)k_4(t_0, t) + k_6(t_0, t)
\end{aligned}\right\}. \tag{2.211}$$

Equation (2.206) is therefore reduced to the following singular integral-differential equation:

$$\begin{aligned}
&\frac{1}{\pi i}\int_\Gamma \Omega_1(t_0, t)\, g'(t)\, dt + \frac{1}{\pi i}\int_\Gamma \Omega_2(t_0, t)\, \overline{g'(t)}\, dt + \\
&+ \frac{1}{2\pi i}\int_\Gamma\left[\frac{\dot{\bar{t}}_0^2(t-t_0)}{\bar{t}-\bar{t}_0} + 1\right]\dot{\bar{t}}^2\,\overline{g'(t)}\,\frac{dt}{t-t_0} + \frac{1}{2\pi i}\times \\
&\times\int_\Gamma\left[\frac{(\bar{t}-\bar{t}_0)}{(t-t_0)} - \dot{\bar{t}}_0^2\right]g'(t)\,\frac{dt}{t-t_0} = 2tD\,\dot{\bar{t}}_0\left[\left(\frac{1}{C_k} + \frac{1}{A_k}\right)\overline{g(t_0)} + \right.
\end{aligned} \tag{2.212}$$

$$+\left(\frac{1}{C_k}-\frac{1}{A_k}\right)g(t_0)\,\dot{\bar{t}}_0^2\Big]-\left[\frac{d}{dt}\left(\frac{\partial w^0}{\partial t}\right)\right]_{t=t_0}. \tag{2.212}$$

Using conjugate magnitudes in (2.212), we obtain

$$\begin{aligned}
&\frac{1}{\pi i}\int_l \overline{\Omega_1(t_0,t)}\,\dot{\bar{t}}^2\,\overline{g'(t)}\,dt+\frac{1}{\pi i}\int_l \overline{\Omega_2(t_0,t)}\,g'(t)\,\dot{\bar{t}}^2\,dt+\\
&+\frac{1}{2\pi i}\int_l\left(\dot{t}_0^2+\frac{t-t_0}{\bar{t}-\bar{t}_0}\right)g'(t)\,\frac{dt}{t-t_0}+\\
&+\frac{1}{2\pi i}\int_l\frac{t-t_0}{\bar{t}-\bar{t}_0}\left(\frac{t-t_0}{\bar{t}-\bar{t}_0}-\dot{t}_0^2\right)\overline{g'(t)}\,\dot{\bar{t}}^2\,\frac{dt}{t-t_0}=\\
&=2iD\,\dot{t}_0\left[\left(\frac{1}{C_k}+\frac{1}{A_k}\right)g(t_0)+\left(\frac{1}{C_k}-\frac{1}{A_k}\right)\overline{g(t_0)}\,\dot{\bar{t}}_0^2\right]+\left[\overline{\frac{d}{dt}\left(\frac{\partial w^0}{\partial t}\right)}\right]_{t=t_0}
\end{aligned} \tag{2.213}$$

We introduce the symbols

$$\left.\begin{aligned}
g(t)&=\varrho_1(t),\quad \overline{g(t)}=\varrho_2(t)\\
g'(t)&=\varrho_1'(t),\quad \overline{g'(t)}\,\dot{\bar{t}}^2=\varrho_2'(t).
\end{aligned}\right\} \tag{2.214}$$

The system of integral-differential equations (2.212) to (2.213) is then written in matrix form:

$$a(t_0)\,\varrho(t_0)+\frac{1}{\pi i}\int_l\frac{K(t_0,t)\,\varrho'(t)\,dt}{t-t_0}=f(t_0). \tag{2.215}$$

Here $\varrho(t)=(\varrho_1,\varrho_2)$ is an unknown vector, $a(t_0)=\|a_{\alpha\beta}(t_0)\|$ and $K(t_0,t)=\|K_{\alpha\beta}(t_0,t)\|$ are known square matrices, and $f(t_0)=(f_1,f_2)$ is a known vector. The elements of the matrices and vectors introduced above are defined as follows:

$$\left.\begin{aligned}
a_{11}(t_0)&=2iD\left(\frac{1}{C}-\frac{1}{A}\right)\dot{\bar{t}}_0^2\\
a_{12}(t_0)&=2iD\left(\frac{1}{C}+\frac{1}{A}\right)\dot{\bar{t}}_0\\
a_{21}(t_0)&=2iD\left(\frac{1}{C}+\frac{1}{A}\right)\dot{\bar{t}}_0\\
a_{22}(t_0)&=2iD\left(\frac{1}{C}-\frac{1}{A}\right)\dot{t}_0\\
K_{11}(t_0,t)&=-\frac{1}{2}\left(\frac{\bar{t}-\bar{t}_0}{t-t_0}-\dot{\bar{t}}_0^2\right)-(t-t_0)\,\Omega_1(t_0,t)\\
K_{12}(t_0,t)&=-\frac{1}{2}\left[1+\frac{\dot{\bar{t}}_0^2(t-t_0)}{\bar{t}-\bar{t}_0}\right]-(t-t_0)\,\Omega_2(t_0,t)\,\dot{\bar{t}}^2\\
K_{21}(t_0,t)&=-\frac{1}{2}\left(1+\frac{t-t_0}{\bar{t}-\bar{t}_0}\dot{\bar{t}}_0^2\right)-(t-t_0)\,\overline{\Omega_2(t_0,t)}\,\dot{\bar{t}}^2\,\dot{\bar{t}}_0^2\\
K_{22}(t_0,t)&=-\frac{1}{2}\left(\frac{t-t_0}{\bar{t}-\bar{t}_0}\,\dot{\bar{t}}_0^2-1\right)\frac{t-t_0}{\bar{t}-\bar{t}_0}-(t-t_0)\,\dot{\bar{t}}_0^2\,\overline{\Omega_1(t_0,t)}
\end{aligned}\right\} \tag{2.216}$$

$$\left.\begin{aligned} f_1(t_0) &= \left[\frac{d}{dt}\left(\frac{\partial w^0}{\partial t}\right)\right]_{t=t_0} \\ -f_2(t_0) &= \left[\frac{d}{dt}\left(\frac{\partial w^0}{\partial t}\right)\right]_{t=t_0}\cdot\dot{t}_0^2 \end{aligned}\right\} \tag{2.216}$$

According to N. P. Vekua /12/, we write

$$\varrho(t) = \int_{t_{0k}}^{t} \varrho'(t_1)\,dt_1 + C_{0k} = \int_{\Gamma} \omega(t, t_1)\varrho'(t_1)\,dt_1 + C_{0k} \tag{2.217}$$
$$(k = 1, 2, \ldots, l),$$

where t_{0k} is a fixed point on γ_k; C_{0k} is an arbitrary constant vector;

$$\omega(t, t_1) = \begin{cases} 1 & \text{if } t_1 \in t_{0k}t, \\ 0 & \text{if } t_1 \bar{\in} t_{0k}t. \end{cases} \tag{2.218}$$

From (2.215) we obtain the following singular integral equation:

$$\frac{1}{\pi i}\int_{\Gamma}\frac{K(t_0, t)\varrho'(t)\,dt}{t - t_0} + \int_{\Gamma} a(t_0)\,\omega(t_0, t)\varrho'(t)\,dt = f(t_0) - a(t_0)C_0, \tag{2.219}$$

where C_0 acquires the values C_{0k} on the contours γ_k.

We introduce the notation

$$K_{\alpha\beta}(t_0, t_0) = B_{\alpha\beta},$$

i. e.,

$$K(t_0, t_0) = \|B_{\alpha\beta}\| = B.$$

In our case

$$B = \begin{Vmatrix} 0 & -1 \\ -1 & 0 \end{Vmatrix}. \tag{2.220}$$

The determinant of this matrix is -1, i. e., it differs from zero everywhere on Γ.

We find from (2.217) that

$$C_{0k} = \varrho(t_{0k}). \tag{2.221}$$

Writing $t_0 = t_{0k}$ in (2.215), we obtain

$$C_{0k} = [a(t_{0k})]^{-1}\left[f(t_{0k}) - \frac{1}{\pi i}\int_{\Gamma}\frac{K(t_{0k}, t)\varrho'(t)\,dt}{t - t_{0k}}\right]. \tag{2.222}$$

Substitution of (2.222) in (2.219) yields

$$\frac{1}{\pi i}\int_{\gamma}\frac{N(t_0, t)\varrho'(t)\,dt}{t-t_0}+\int_{\gamma}a(t_0)\,\omega(t_0, t)\varrho'(t)\,dt=f(t_0)-a(t_0)C_*, \tag{2.223}$$

where $N(t_0, t)$ and C_* (C_{*1}, C_{*2}) are known functions which on the contours γ_k acquire the values

$$\left.\begin{aligned} N_k(t_0, t)&=K(t_0, t)-a(t_0)[a(t_{*k})]^{-1}K(t_{*k}, t)\frac{t-t_0}{t-t_{*k}}\\ C_{*k}&=[a(t_{*k})]^{-1}f(t_{*k})\end{aligned}\right\}, \tag{2.224}$$

respectively. We rewrite equation (2.223) as follows:

$$\Lambda\varrho'=\frac{B}{\pi i}\int_{\gamma}\frac{\varrho'(t)\,dt}{t-t_0}+\frac{1}{\pi i}\int_{\gamma}N_0(t_0, t)\varrho'(t)\,dt=f(t_0)-a(t_0)C_*, \tag{2.225}$$

where $N_0(t_0t)$ is a known matrix which on γ_k acquires the values

$$\begin{aligned} N_{0k}(t_0, t)=\frac{N(t_0, t)-B}{t-t_0}+\pi i a(t_0)\,\omega(t_0, t)=K_0(t_0, t)-\\ -a(t_0)[a(t_*)]^{-1}\frac{K(t_*, t)}{t-t_*},\end{aligned} \tag{2.226}$$

with

$$\left.\begin{aligned} K_0(t_0, t)&=\|K_{0\alpha\beta}(t_0, t)\|\\ K_{011}(t_0, t)&=-\frac{1}{2(t-t_0)}\left(\frac{\bar{t}-\bar{t}_0}{t-t_0}-\dot{\bar{t}}_0^2\right)-\\ &-2\pi D\left(\frac{1}{C}-\frac{1}{A}\right)\dot{\bar{t}}_0^3\omega(t_0, t)-\Omega_1(t_0, t)\\ K_{012}(t_0, t)&=-\frac{1}{2(t-t_0)}\left(\frac{t-t_0}{\bar{t}-\bar{t}_0}\dot{\bar{t}}_0^2-1\right)-\\ &-2\pi D\left(\frac{1}{C}+\frac{1}{A}\right)\dot{\bar{t}}_0\,\omega(t_0, t)-\dot{t}^2\Omega_2(t_0, t)\\ K_{021}(t_0, t)&=-\frac{1}{2(t-t_0)}\left(\frac{t-t_0}{\bar{t}-\bar{t}_0}\dot{\bar{t}}_0^2-1\right)-\\ &-2\pi D\left(\frac{1}{C}+\frac{1}{A}\right)\dot{\bar{t}}_0\,\omega(t_0, t)-\dot{\bar{t}}_0^2\dot{\bar{t}}^2\overline{\Omega_2(t_0, t)}\\ K_{022}(t_0, t)&=-\frac{1}{2(\bar{t}-\bar{t}_0)}\left(\frac{t-t_0}{\bar{t}-\bar{t}_0}\dot{\bar{t}}_0^2-1\right)-\\ &-2\pi D\left(\frac{1}{C}-\frac{1}{A}\right)\dot{t}_0\,\omega(t_0, t)-\dot{\bar{t}}_0^2\overline{\Omega_1(t_0, t)}\end{aligned}\right\} \tag{2.227}$$

The characteristic equation corresponding to (2.225) is:

$$M\varrho' = \frac{B}{\pi i}\int_\gamma \varrho'(t)\frac{dt}{t-t_0} = f_*(t_0). \tag{2.228}$$

Because of (2.220), the characteristic system (2.228) consists in our case of separate equations

$$\left.\begin{aligned} -\frac{1}{\pi i}\int_\gamma \frac{\varrho_2'(t)\,dt}{t-t_0} &= f_{*1}(t_0)\\ -\frac{1}{\pi i}\int_\gamma \frac{\varrho_1'(t)\,dt}{t-t_0} &= f_{*2}(t_0)\end{aligned}\right\}. \tag{2.229}$$

The indices of equations (2.229) are equal to zero, and therefore these equations can be solved for any right-hand side $f_{*i}(t_0)$ $(i = 1, 2)$. Their solution is given by the inversion formula for Cauchy-type integrals (/81/, p. 138):

$$\varrho'(t_0) = \frac{B}{\pi i}\int_\gamma \frac{f_*(t)\,dt}{t-t_0}. \tag{2.230}$$

This is the general solution of (2.225), with

$$f_*(t_0) = f(t_0) - a(t_0)C_* - \frac{1}{\pi i}\int_\gamma N_0(t_0, t)\varrho'(t)\,dt. \tag{2.231}$$

Substituting (2.231) in (2.230), we obtain the following Fredholm integral equation of the second kind, which is equivalent to (2.215):

$$\varrho'(t_0) + \frac{1}{\pi i}\int_\gamma K^*(t_0, t)\varrho'(t)\,dt = F^*(t_0) + \gamma(t_0)C_*, \tag{2.232}$$

where

$$\left.\begin{aligned} \gamma(t_0) &= -\frac{B}{\pi i}\int_\gamma \frac{a(t)\,dt}{t-t_0}\\ F^*(t_0) &= \frac{B}{\pi i}\int_\gamma \frac{f(t)\,dt}{t-t_0}\\ K^*(t_0, t) &= \frac{B}{\pi i}\int_\gamma \frac{N_0(t_1, t)\,dt_1}{t_1-t_0}\end{aligned}\right\}. \tag{2.233}$$

The functions $a_{\alpha\beta}(t)$ are defined by (2.216).

Fredholm's alternative is applicable to equation (2.232). To prove that it can be solved it suffices therefore to show that the corresponding homogeneous equation, obtained from (2.232) for

$$F^*(t_0) + \gamma(t_0)C_* = 0 \tag{2.234}$$

has no nontrivial solutions. By (2.220) and (2.233), we can rewrite (2.234) in the form

$$\left.\begin{aligned}\int_\Gamma [f_2(t) - a_{21}C_{\cdot 1} - a_{22}C_{\cdot 2}]\frac{dt}{t-t_0} = 0\\ \int_\Gamma [f_1(t) - a_{11}C_{\cdot 1} - a_{12}C_{\cdot 2}]\frac{dt}{t-t_0} = 0\end{aligned}\right\}. \tag{2.235}$$

Solving system (2.235) by the inversion formula for Cauchy-type integrals, we find

$$\left.\begin{aligned}f_1(t) = a_{11}C_{\cdot 1} + a_{12}C_{\cdot 2}\\ f_2(t) = a_{21}C_{\cdot 1} + a_{22}C_{\cdot 2}\end{aligned}\right\}. \tag{2.236}$$

If we take into account expressions (2.216) for the functions f_1, f_2, $a_{\alpha\beta}$ and (2.224) for the vector $C_{\cdot}(C_{\cdot 1}, C_{\cdot 2})$, it is found that $C_{\cdot 1} = C_{\cdot 2}$ and that both equations (2.236) coincide. It therefore suffices to consider the first which can be written in the following form, identical with (2.185):

$$\frac{d}{ds}\left(\frac{\partial w^0}{\partial t}\right) = 2iD\left[\left(\frac{1}{C} - \frac{1}{A}\right)\dot{t}^2 C_{\cdot 1} + \left(\frac{1}{C} + \frac{1}{A}\right)\bar{C}_{\cdot 1}\right]. \tag{2.237}$$

Let now $\varrho_{\cdot}'(t)$ be any solution of the homogeneous equation corresponding to (2.232), and let $g_{\cdot}(t)$, $\varphi_{\cdot}(z)$, $\psi_{\cdot}(z)$ be the corresponding values of the functions $g(t)$, $\varphi(z)$, $\psi(z)$ defined by (2.214), (2.203), and (2.157), when $\varrho'(t)$ is replaced by $\varrho_{\cdot}'(t)$. The functions $g_{\cdot}(t)$, $\varphi_{\cdot}(z)$ and $\psi_{\cdot}(z)$ satisfy the boundary conditions (2.147) to (2.150) when (2.237) holds, and therefore form the solution of our problem under condition (2.237); on the basis of the uniqueness theorem (cf. note 3 at the end of § 9) we then obtain from (2.187)

$$g_{\cdot}(t) = \text{const.}$$

Hence $g_{\cdot}'(t) = 0$, and according to (2.214),

$$\varrho_{\cdot}'(t) = 0 \text{ everywhere on } \Gamma. \tag{2.238}$$

The homogeneous equation corresponding to (2.232) thus has no nontrivial solutions and (2.232) has therefore always one and only one solution $\varrho'(t)$.

Substituting this value $\varrho'(t)$ in (2.215), we obtain the vector $\varrho(t)$, and thus the function $g(t)$ by means of (2.214). Knowing $g(t)$, we determine from (2.203) $\varphi^0(z)$ except for the constant α_0, and $\psi^0(z)$ except for the constant $\alpha_0' = \varkappa\bar{\alpha}_0$. We then find the piecewise-holomorphic functions $\varphi(z)$ and $\psi(z)$ from (2.157). The problem is thus solved.

Equation (2.232) can be solved by approximations /3, 48, 69, 79, 132/. If the method of successive approximations is used, we obtain convergence in the mean, the generalized limit of these approximations giving a unique solution of (2.232) (/79/, § 13).

N o t e. The problem is solved similarly when condition (2.160) of the second basic problem is specified on the external contour L of the circular plate.

The functions $\varphi^0(z)$ and $\psi^0(z)$ are then given by (2.203), where $\varkappa = -1$ and $\alpha_0' = -\bar{\alpha}_0$. We must also replace $\varkappa$ by -1 in (2.209). The solution is identical to the preceding in all other respects.

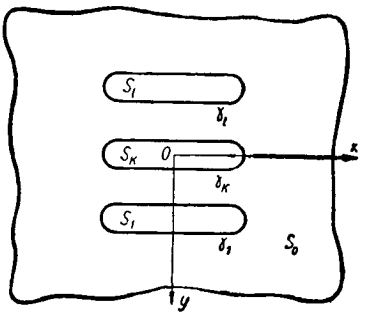


FIGURE 9.

Case of an infinite domain. Consider the particular case when the middle plane of the plate, reinforced by stiffening ribs (Figure 9), occupies the entire xy-plane. The domain S_0 is infinite and l-fold connected. Boundary condition (2.147) disappears in this case. The functions $\varphi^0(z)$ and $\psi^0(z)$, appearing in (2.157), are holomorphic in the entire plane, including the point at infinity, and we therefore find from Liouville's theorem (/104/, p. 209):

$$\varphi^0(z) = \alpha_0 = \text{const}, \quad \psi^0(z) = \alpha_0 = \text{const}.$$

According to (2.207) and (2.210), we obtain

$$\Omega(t_0) = \Omega_1(t_0 t) = \Omega_2(t_0 t) = 0. \tag{2.239}$$

The expressions for $K_{\alpha\beta}(t_0, t)$ in (2.216) and for $K_{0\alpha\beta}(t_0, t)$ in (2.227) are correspondingly simplified.

Thus, in this particular case the problem is reduced to the same Fredholm integral equation (2.232), but with a simpler kernel.

§11. FIRST BASIC PROBLEM FOR A CIRCULAR PLATE WITH A CONCENTRIC STIFFENING RIB

A circular plate with a concentric rib (cf. Figure 10) is subjected to an arbitrary transverse load and arbitrary forces and moments along the contour $r = R_1$.

We determine the additional state of stress, caused by the stiffening rib, by representing on Γ the unknown single-valued function $g(t)$, defined by (2.151), in the form

$$g(t) = \sum_{-\infty}^{+\infty} a_n \frac{t^n}{R^n}. \tag{2.240}$$

where a_n are unknown complex constants.

Substitution of this expression in (2.156) yields

$$h(t) = \sum_{-\infty}^{+\infty} [\bar{a}_{-n} + (n+2)\, a_{n+2}] \frac{t^n}{R^n}. \tag{2.241}$$

Substituting (2.240) and (2.241) in (2.157) for $|z| < R$, we find for the domain S_1:

$$\begin{gathered} \varphi_1(z) = \varphi^0(z) + \sum_{n=0}^{\infty} a_n \frac{z^n}{R^n}, \\ (|z| < R) \\ \psi_1(z) = \psi^0(z) - \\ - \sum_{n=0}^{\infty} [\bar{a}_{-n} + (n+2)\, a_{n+2}] \frac{z^n}{R_n} \end{gathered} \tag{2.242}$$

(The contour Γ in Figure 10 is described in opposite sense to that in Figure 8, in § 10). Substitution of these functions in (2.150) yields the following boundary condition for $\varphi^0(z)$ and $\psi^0(z)$:

$$\overline{\varphi^0(t)} + t\bar{\varphi}^{0\prime}(t) + \psi^0(t) = P(t) - R(t) + \sum_0^{\infty} \bar{a}_{-n} \frac{t^n}{R^n} - \sum_2^{\infty} \bar{a}_n \frac{R^n}{t^n} - \bar{a}_0 - (a_1 + \bar{a}_1) \frac{R}{t}. \tag{2.243}$$

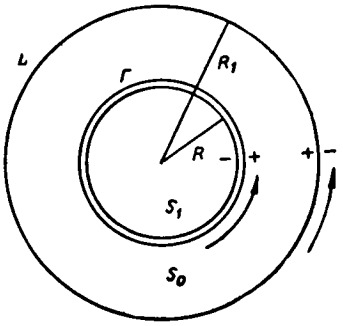


FIGURE 10.

The function $P(t)$ here is expressed according to (2.182) and (2.240) in the form

$$P(t) = K \ln \sigma + 2RD \sum_{-\infty}^{+\infty} B_n \sigma^n, \tag{2.244}$$

where

$$\sigma = e^{i\theta},$$

$$\left.\begin{aligned} K &= 2RD\left[\bar{a}_0\left(\frac{1}{C}+\frac{1}{A}\right)-a_2\left(\frac{1}{C}-\frac{1}{A}\right)\right] \\ B_0 &= \left(\frac{1}{C}+\frac{1}{A}\right)\sum_{-\infty}^{+\infty}{}^{\bullet}\frac{\bar{a}_n}{n}+\left(\frac{1}{C}-\frac{1}{A}\right)\sum_{-\infty}^{+\infty}{}'\frac{a_n}{n-2} \\ B_n &= \left(\frac{1}{C}+\frac{1}{A}\right)\frac{\bar{a}_{-n}}{n}-\left(\frac{1}{C}-\frac{1}{A}\right)\frac{a_{n+2}}{n} \\ &(n=\pm 1,\pm 2,\ldots). \end{aligned}\right\} \tag{2.245}$$

The sums $\sum_{-\infty}^{+\infty}{}^{\bullet}$ and $\sum_{-\infty}^{+\infty}{}'$ contain no terms corresponding, respectively, to $n=0$ and $n=2$.

Since the functions on the left-hand side of (2.243) are single-valued, the function $P(t)$, given by (2.244), is also single-valued. Hence, $K=0$, i.e.,

$$\bar{a}_0\left(\frac{1}{C}+\frac{1}{A}\right)-a_2\left(\frac{1}{C}-\frac{1}{A}\right)=0. \tag{2.246}$$

We expand the function $R(t)$, defined by (2.152,a), in a Fourier series:

$$R(t)=\frac{\partial w_1^0}{\partial t}-\frac{C_{31}}{2i}=-2\sum_{-\infty}^{+\infty}b_n\sigma^n-\frac{1}{2i}C_{31}, \tag{2.247}$$

where b_n are known complex constants.

Substitution of (2.244) and (2.247) in (2.243) yields

$$\overline{\varphi^0(t)}+\bar{t}\,\varphi^{0\prime}(t)+\psi^0(t)=\sum_{-\infty}^{+\infty}d_n\sigma^n \quad \text{on } \Gamma, \tag{2.248}$$

where

$$\left.\begin{aligned} d_{-n} &= 2(RDB_{-n}+b_{-n})-\bar{a}_n \quad (n=2,3,4,\ldots) \\ d_n &= 2(RDB_n+b_n)+\bar{a}_{-n} \quad (n=1,2,\ldots) \\ d_{-1} &= 2(RDB_{-1}+b_{-1})-(a_1+\bar{a}_1) \\ d_0 &= 2(RDB_0+b_0)+C_{31}/2i \end{aligned}\right\}. \tag{2.249}$$

The unknown functions $\varphi^0(z)$ and $\psi^0(z)$ are holomorphic within the circle $|z|\leqslant R_1$. We represent them in the form

$$\varphi^0(z)=\sum_{n=0}^{\infty}\alpha_n\frac{z^n}{R^n}, \quad \psi^0(z)=\sum_{n=0}^{\infty}\alpha_n'\frac{z^n}{R^n}. \tag{2.250}$$

To determine the functions (2.250) we must thus solve the second fundamental bending problem for the circular domain S_1, with the boundary condition (2.248).

Solving this problem by Muskhelishvili's method (/82/, § 80), we find

$$\left.\begin{aligned}\varphi^0(z) &= \sum_0^\infty \bar{d}_{-n}\frac{z^n}{R^n} - \frac{\bar{\alpha}_1 z}{R} - 2\bar{\alpha}_2 - \bar{\alpha}_0'\\ \psi^0(z) &= \sum_0^\infty [d_n - (n+2)\bar{d}_{-(n+2)}]\frac{z^n}{R^n} + \\ &+ (\alpha_1 + \bar{\alpha}_1 - \bar{d}_{-1})\frac{R}{z} - \bar{\alpha}_0\end{aligned}\right\}. \tag{2.251}$$

We now require that the constants α_0, α_0', α_1, and α_2 in (2.251) represent the coefficients of z^0, z^1 and z^2 in (2.250). We therefore set $z = 0$ in (2.251) and in the expressions obtained from the first equation (2.251) by differentiating once and twice with respect to z and obtain [cf. (2.191), (2.194)]:

$$\left.\begin{aligned}\alpha_0 &= \bar{d}_0 - 2\bar{\alpha}_2 - \bar{\alpha}_0'\\ \alpha_1 &= \bar{d}_{-1} - \bar{\alpha}_1, \quad \alpha_2 = \bar{d}_{-2}\end{aligned}\right\}. \tag{2.252}$$

Substitution of these equations and of (2.249) in (2.251) yields

$$\left.\begin{aligned}\varphi^0(z) &= \alpha_1\frac{z}{R} + \sum_{n=2}^\infty [2(RD\bar{B}_{-n} + \bar{b}_{-n}) - a_n]\frac{z^n}{R^n} + \alpha_0\\ \psi^0(z) &= \sum_{n=1}^\infty [2(RDB_n + b_n) + \bar{a}_{-n} + (n+2)a_{n+2} - \\ &- 2(n+2)(RD\bar{B}_{-(n+2)} + b_{-(n+2)})]\frac{z^n}{R^n} + \alpha_0'\end{aligned}\right\}. \tag{2.253}$$

The coefficients b_n, defined by (2.247), are known, and the coefficients B_n are expressed through the unknown coefficients a_n with the aid of (2.245).

The functions $\varphi^0(z)$ and $\psi^0(z)$, given by (2.253), for domain S_1, are analytically continued across Γ onto the entire domain $S = S_1 + S_0$; we can therefore insert them into (2.157) and determine the functions $\varphi(z)$ and $\psi(z)$ for the domain S_0, i.e., for $R < |z| < R_1$.

By calculating the corresponding Cauchy-type integrals we obtain finally [cf. (2.242)]

$$\left.\begin{aligned}\varphi_0(z) &= \varphi^0(z) - \sum_1^\infty a_{-n}\frac{R^n}{z^n}\\ \psi_0(z) &= \psi^0(z) + \sum_1^\infty [\bar{a}_n + (2-n)a_{2-n}]\frac{R^n}{z^n}\end{aligned}\right\}. \tag{2.254}$$

Substituting (2.254) in the boundary condition (2.147), with $m = 0, z = R_1\sigma$, and $C_{1,1} = C_{2,1} = 0$, and equating the coefficients of equal powers of σ, we

obtain the following system of algebraic equations:

$$Im a_1 = 0, \tag{2.255}$$

$$\frac{1-\varkappa}{k} a_1 + k(a_1 + \bar{a}_1) = 0, \tag{2.256}$$

$$-\varkappa\left[2\bar{b}_{-2} - a_2\left(1 + \frac{1}{\delta_1} + \frac{1}{\delta_2}\right) + \bar{a}_0\left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right)\right] + k^4 a_2 = 0, \tag{2.257}$$

$$\frac{2}{k^2}\left[\bar{a}_0\left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right) - a_2\left(1 + \frac{1}{\delta_1} + \frac{1}{\delta_2}\right) + 2\bar{b}_{-2}\right] = \varkappa\bar{a}_0 - a_0', \tag{2.258}$$

$$\left.\begin{aligned} a_{-n}\left[\frac{n}{\varkappa}(1-k^2)k^{2(n+1)} - \frac{2}{n+2}\left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right)\right] + \\ + \bar{a}_{n+2}\left[1 + \frac{1}{\varkappa}k^{2(n+2)} + \frac{2}{n+2}\left(\frac{1}{\delta_2} + \frac{1}{\delta_1}\right)\right] = 2b_{-(n+2)} \\ a_{-n}\left[(1+\varkappa k^{2n}) + \frac{2}{n}\left(\frac{1}{\delta_2} + \frac{1}{\delta_1}\right) + \frac{n}{\varkappa}(n+2)(1-k^2)^2 k^{2n}\right] + \\ + \bar{a}_{n+2}\left[-\frac{n+2}{\varkappa}(k^2-1)k^{2(n+1)} - \frac{2}{n}\left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right)\right] = \\ = -2\bar{b}_n \quad (n = 1, 2, 3, \ldots) \end{aligned}\right\}, \tag{2.259}$$

where

$$\delta_1 = \frac{A}{RD}, \quad \delta_2 = \frac{C}{RD}, \quad k = \frac{R}{R_1}. \tag{2.260}$$

We shall now show that the coefficient b_{-1} in (2.247) is a real number. In fact, by Fourier's formula,

$$b_{-1} = -\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\cdot\frac{\partial w_1^0}{\partial t}e^{i\theta}d\theta = \frac{i}{4\pi}\int_0^{2\pi}\frac{\partial w_1^0}{\partial t}id\theta = \frac{1}{8\pi}\int_0^{2\pi}\left(\frac{\partial w_1^0}{\partial n} + i\frac{\partial w_1^0}{\partial s}\right)d\theta,$$

whence

$$\mathrm{Im} b_{-1} = \frac{1}{8\pi R}\int_0^{2\pi}\frac{\partial w_1^0}{\partial\theta}d\theta = 0 \tag{2.261}$$

since the deflections w^0 are single-valued.

Taking into account (2.261), (2.255), (2.245), and (2.249), we obtain from the second equation (2.252)

$$\left.\begin{aligned} &\mathrm{Im} a_1 = 0 \\ &a_1 = b_{-1} - a_1\left(1 + \frac{2}{\delta_1}\right) \end{aligned}\right\}. \tag{2.262}$$

Solving simultaneously (2.262) and (2.256), we find

$$a_1 = \frac{2k^2 a_1}{\varkappa - 1}, \quad a_1 = \frac{\delta_1(\varkappa - 1)b_{-1}}{2(\varkappa - 1) + \delta_1(2k^2 + \varkappa - 1)}. \tag{2.263}$$

In view of the arbitrariness of the function $\varphi_0(z)$ (cf. (2.174) and (2.254)), we assume

$$a_0 = 0. \tag{2.264}$$

We than find from (2.257), (2.258), and (2.246):

$$\left.\begin{aligned} \bar{a}_0 &= \frac{2\varkappa(\delta_1-\delta_2)\,b_{-2}}{4\varkappa+(\varkappa+k^4)(\delta_1+\delta_2)} \\ a_2 &= \frac{2\varkappa(\delta_1+\delta_2)\,\bar{b}_{-2}}{4\varkappa+(\varkappa+k^4)(\delta_1+\delta_2)} \\ a_0' &= -\frac{4k^2(\delta_1+\delta_2)\,\bar{b}_{-2}}{4\varkappa+(\varkappa+k^4)(\delta_1+\delta_2)} \end{aligned}\right\}. \tag{2.265}$$

The first equation (2.252) serves for obtaining the constant C_{31} entering in the coefficient d_0 given by (2.249). We do not require this constant, and shall not determine it. Finally, the system (2.259) yields the unknown coefficients a_{-n}, a_{n+2} $(n = 1, 2, 3, \ldots)$:

$$\left.\begin{aligned} a_{-n} &= \frac{2}{\Delta_n}\left\{\left[\frac{1}{\varkappa}(n+2)(k^2-1)\,k^{2n+2}\delta_1\delta_2+\frac{2}{n}(\delta_1-\delta_2)\right]b_{-(n+2)}-\right. \\ &\quad \left.-\left[\left(1+\frac{1}{\varkappa}k^{2n+4}\right)\delta_1\delta_2+\frac{2}{n+2}(\delta_1+\delta_2)\right]\bar{b}_n\right\} \\ a_{n+2} &= \frac{2}{\Delta_n}\left\{\left[\frac{k^{2n}}{\varkappa}(1-k^2)(n+2)\,n\delta_1\delta_2+\right.\right. \\ &\quad \left.+\frac{2}{n}(\delta_1+\delta_2)+(1+\varkappa k^{2n})\,\delta_1\delta_2\right]\cdot 2\bar{b}_{-(n+2)}+ \\ &\quad \left.+\left[\frac{n}{\varkappa}k^{2n+2}(1-k^2)\,\delta_1\delta_2-\frac{2}{n+2}(\delta_1-\delta_2)\right]b_n\right\} \end{aligned}\right\}, \tag{2.266}$$

where

$$\begin{aligned} \Delta_n &= \frac{16}{n(n+2)}+2\delta_1\left\{\frac{\varkappa k^{2n}}{n+2}+\frac{2(n+1)}{n(n+2)}+\frac{nk^{2n}}{\varkappa}\left[\frac{k^2}{n}+1-k^2\right]^2\right\}+ \\ &\quad +2\delta_2\left\{\frac{\varkappa k^{2n}}{n+2}+\frac{2(n+1)}{n(n+2)}+\frac{nk^{2n}}{\varkappa}\left[\frac{k^2}{n}-1+k^2\right]^2\right\}+ \\ &\quad +\delta_1\delta_2\left\{\varkappa k^{2n}+(1+k^{4n+4})+\frac{k^{2n}}{\varkappa}[k^4+n(n+2)(1-k^2)^2]\right\}. \end{aligned} \tag{2.267}$$

The problem stated has thus been completely solved. All the coefficients of the function $g(t)$, defined by (2.240), are given by (2.263), (2.265), and (2.266), while α_0, a_0 and a_1 are given by (2.264), (2.265), and (2.263), respectively. The functions $\varphi^0(z)$, $\psi^0(z)$ ((2,253)), $\varphi_0(z)$, $\psi_0(z)$ ((2.254)), and $\varphi_1(z)$, $\psi_1(z)$ ((2.242)) are thus determined.

Writing $k = 1$, we obtain the solution for the case of a thin stiffening rib along the edge of a circular plate.

§ 12. AXISYMMETRICAL BENDING OF A SIMPLY-SUPPORTED CIRCULAR PLATE WITH A STIFFENING RIB

The solution obtained in the preceding section is considerably simplified in the particular case of axisymmetrical bending. For an arbitrary axisymmetrical load (Figure 11) we have in this case on Γ,

$$\left(\frac{\partial w^0}{\partial t}\right)_\Gamma = \frac{1}{2}\left(\frac{dw^0}{dr}\right)_\Gamma \sigma^{-1}. \tag{2.268}$$

hence all coefficients b_n in (2.247) vanish except

$$b_{-1} = -\frac{1}{4}\left(\frac{dw^0}{dr}\right)_\Gamma \tag{2.269}$$

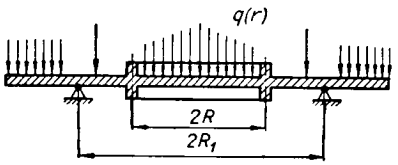


FIGURE 11.

Thus, by (2.263)

$$\left.\begin{aligned} a_1 &= -\frac{1}{4}\frac{\delta_1(1+\nu)}{2(1+\nu)+\delta_1[k^2(1-\nu)+(1+\nu)]}\left(\frac{dw^0}{dr}\right)_\Gamma \\ \alpha_1 &= -\frac{k^2}{4}\frac{\delta_1(1-\nu)}{2(1+\nu)+\delta_1[k^2(1-\nu)+(1+\nu)]}\left(\frac{dw^0}{dr}\right)_\Gamma \end{aligned}\right\}. \tag{2.270}$$

All the remaining coefficients a_n and α_0' vanish. Expressions (2.242) and (2.254) reduce, respectively, to

$$\varphi_1(z) = \frac{k^2(1-\nu)+(1+\nu)}{1+\nu} a_1 \frac{z}{R}, \quad \psi_1(z) = 0,$$

$$\varphi_0(z) = \frac{k^2(1-\nu)}{1+\nu} a_1 \frac{z}{R}, \quad \psi_0(z) = 2a_1 \frac{R}{z}.$$

The corresponding additional deflections are [cf. (2.142)]

$$w_k^1 = 2\,\mathrm{Re}[\bar{z}\varphi_k(z) + \chi_k(z)] \quad (k = 0, 1). \tag{2.271}$$

Using the condition of arbitrariness (2.174), we find

$$w^1 = \mu_1(r, k)\lambda_1(\delta_1, k) R_1 \left(\frac{dw^0}{dr}\right)_\Gamma + C^*, \tag{2.272}$$

where,

$$\left.\begin{aligned}\mu_{10}(r)&=\frac{2(1-\nu)}{1+\nu}\left(1-\frac{r^2}{R_1^2}\right)-4\ln\frac{r}{R_1}\quad\text{for } R\leqslant r\leqslant R_1\\ \mu_{11}(r,k)&=\frac{2(1-\nu)}{1+\nu}\left(1-\frac{r^2}{R_1^2}\right)+2\left(1-\frac{r^2}{k^2R_1^2}-2\ln k\right)\\ &\text{for } 0\leqslant r\leqslant R\end{aligned}\right\}, \tag{2.273}$$

and

$$\lambda_1(\delta_1,k)=\frac{k}{4}\cdot\frac{\delta_1(1+\nu)}{2(1+\nu)+\delta_1[k^2(1-\nu)+(1+\nu)]}. \tag{2.274}$$

The arbitrary constant C^* is determined from the condition that the deflections w^1 vanish on the supporting contour.

Thus, irrespective of the specific form of the axisymmetrical load on the plate, the additional deflections (except for the constant C^*), and hence also the additional bending moments in the plate, due to the rib, are always proportional to $R_1\left(\frac{dw^0}{dr}\right)_\Gamma$. We can therefore calculate once for all the functions $\mu_1(r,k)$ and $\lambda_1(\delta_1,k)$ for any case of axisymmetrical bending. The factor $R_1\left(\frac{dw^0}{dr}\right)_\Gamma$ in (2.272) is determined in each case as a function of the plate size and the load, by the formulas for a nonreinforced circular plate /9, 75, 136/.

The functions $\mu_1(r,k)$ and $\lambda_1(\delta_1,k)$ are given in Tables 1 and 2 for $\nu=0.3$. Since λ_1 and μ_1 are always positive, the additional deflections w^1 have the same sign as $\left(\frac{dw^0}{dr}\right)_\Gamma$.

Similarly, in order to determine the influence of the stiffening rib on the bending moments M_r and M_θ, we obtain the additional moments M_{rk}^1 and $M_{\theta k}^1$ from (1.19) and (2.272):

For $(R\leqslant r\leqslant R_1)$:

$$M_{r0}^1=\frac{4D(1-\nu)}{R_1}\left(\frac{dw^0}{dr}\right)_\Gamma\cdot\lambda_1\cdot\left(1-\frac{R_1^2}{r^2}\right), \tag{2.275}$$

$$M_{\theta 0}^1=\frac{4D(1-\nu)}{R_1}\left(\frac{dw^0}{dr}\right)_\Gamma\cdot\lambda_1\cdot\left(1+\frac{R_1^2}{r^2}\right). \tag{2.276}$$

For $r\leqslant R$:

$$M_{r1}^1=M_{\theta 1}^1=\frac{4D}{R_1}\left(\frac{dw^0}{dr}\right)_\Gamma\cdot\lambda_1\cdot\left[(1-\nu)+\frac{1+\nu}{k^2}\right]. \tag{2.277}$$

The additional moments M_{r0}^1 and $M_{\theta 0}^1$ are thus found for $R\leqslant r\leqslant R_1$ by multiplying the functions $\left(1-\frac{R_1^2}{r^2}\right)$ and $\left(1+\frac{R_1^2}{r_2}\right)$ by λ_1 (cf. Table 2) and by $\frac{4D(1-\nu)}{R_1}\left(\frac{dw^0}{dr}\right)_\Gamma$ which depend on the dimensions of the given plate and on the load.

The additional bending moments in the central region $0\leqslant r\leqslant R$ are constant, being given by (2.277).

Obviously, for $k=0$ (rib reduced to a point) we obtain $w^1=C^*=0$, since then $\left(\frac{dw^0}{dr}\right)_\Gamma=0$.

TABLE 1

r/R_1	k				
	0.2	0.4	0.6	0.8	1.0
	$\mu_1(r,k)$				
0	9.515	6.742	5.120	3.969	3.077
0.2	7.472	6.199	4.855	3.804	2.954
0.4	4.570	4.570	4.059	3.297	2.585
0.6	2.733	2.733	2.733	2.457	1.969
0.8	1.280	1.280	1.280	1.280	1.108
1.0	0	0	0	0	0

TABLE 2

δ_1	k				
	0.2	0.4	0.6	0.8	1.0
	$\lambda_1(\delta_1,k)$				
0.5	0.00996	0.01966	0.02888	0.03747	0.04514
1	0.01655	0.03240	0.04697	0.05980	0.07652
2	0.02473	0.04794	0.06837	0.08530	0.09848
5	0.03517	0.06832	0.09411	0.11464	0.12897
10	0.04093	0.07775	0.10761	0.12948	0.14381
∞	0.04895	0.09207	0.12564	0.14874	0.16250

Equations (2.272) and (2.275) to (2.277) enable us to calculate the additional deflections and moments, due to the stiffening rib, in a circular plate subjected to an arbitrary axisymmetrical load. *

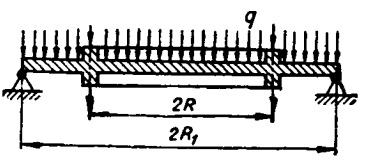


FIGURE 12.

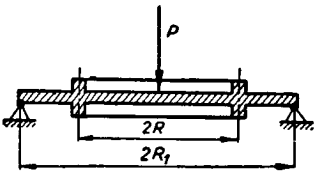


FIGURE 13.

In particular, if $k = 1$ the stiffening rib is located at the edge of the circular plate. We then obtain from (2.272) the additional deflection as

$$w^1 = \frac{R_1\delta_1}{2(1+\nu+\delta_1)}\left(1-\frac{r^2}{R_1^2}\right)\left(\frac{dw^0}{dr}\right)_\Gamma + C^*. \qquad (2.278)$$

Letting in (2.274) $\delta_1 \to \infty$, we obtain the function $\lambda_1(\delta_1, k)$ for the case when the center of the plate is formed by an absolutely rigid core of radius R. The additional deflection w^1 is then

$$w^1 = \frac{1}{4}\mu_1(r,k)R\left(\frac{dw^0}{dr}\right)_\Gamma \frac{1+\nu}{k^2(1-\nu)+(1+\nu)} + C^*. \qquad (2.279)$$

* Several problems of the axisymmetrical bending of circular and annular plates with one or two annular ribs have been solved by Grach /31/ (cf. also § 8 of Chapter IV, and §§ 4, 11 of Chapter III).

We shall consider several examples.

1) Circular plate with concentric rib, subjected to a uniformly distributed load q = const. (Figure 12). In this case (/136/, Chapter III)

$$w^0 = \frac{qR_1^4}{64(1+\nu)D}\left(1-\frac{r^2}{R_1^2}\right)\left[(5+\nu)-\frac{r^2}{R_1^2}(1+\nu)\right]. \qquad (2.280)$$

Substituting (2.280) in (2.272), with $C^* = 0$, we find the additional deflection

$$w^1 = \mu_1(r,k)\,\lambda_1(\delta_1,k)\,\frac{qR_1^4 k}{16(1+\nu)D}\,[(1+\nu)k^2-(3+\nu)],$$

which, added to the basic deflection (2.280), gives the total deflection $w(r)$. For the particular case $k=1$, when the stiffening rib is located on the plate edge, the total deflection is

$$w = \frac{qR_1^4}{64D}\left[\frac{r^4}{R_1^4} - 2\,\frac{3+\nu+\delta_1}{1+\nu+\delta_1}\cdot\frac{r^2}{R_1^2} + \frac{5+\nu+\delta_1}{1+\nu+\delta_1}\right]. \qquad (2.281)$$

2) Circular plate with concentric rib, loaded at the center by a force P (Figure 13). The basic deflection w^0 is

$$w^0 = \frac{PR_1^2}{16\pi D}\left[\frac{3+\nu}{1+\nu}\left(1-\frac{r^2}{R_1^2}\right)+2\,\frac{r^2}{R_1^2}\ln\frac{r}{R_1}\right]. \qquad (2.282)$$

The additional deflection w^1 is determined from (2.272) with $C^* = 0$:

$$w^1 = \mu_1(r,k)\,\lambda_1(\delta_1,k)\left(\ln k - \frac{1}{1+\nu}\right)\frac{PR_1^2 k}{4\pi D}. \qquad (2.283)$$

Adding (2.283) and (2.282), we find the total deflection $w(r)$. If the stiffening rib rests on a support ($k=1$), the total deflection of the plate is

$$w = \frac{PR_1^2}{16\pi D}\left[\left(1-\frac{r^2}{R_1^2}\right)\frac{3+\nu+\delta_1}{1+\nu+\delta_1} + 2\,\frac{r^2}{R_1^2}\ln\frac{r}{R_1}\right]. \qquad (2.284)$$

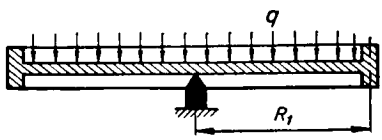


FIGURE 14.

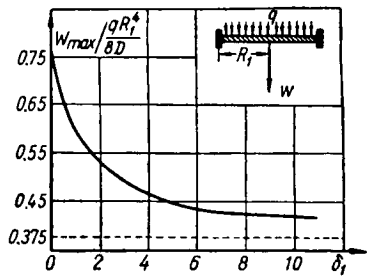


FIGURE 15.

3) Uniformly loaded circular plate with reinforced edge. The plate is supported at the center and reinforced by a thin ring along the contour $r = R_1$ (Figure 14). The basic deflection w^0 for a nonstiffened plate is

$$w^0 = \frac{qR_1^4}{8D}\left[\frac{3+\nu}{4(1+\nu)}\cdot\frac{r^2}{R_1^2} + \frac{1}{8}\cdot\frac{r^4}{R_1^4} - \frac{r^2}{R_1^2}\ln\frac{r}{R_1}\right]. \tag{2.285}$$

In this case the constant C^* in (2.272) is determined from the condition $w^1 = 0$ at $r = 0$:

$$C^* = \frac{-R_1\delta_1}{2(1+\nu+\delta_1)}\left(\frac{dw^0}{dr}\right)_\Gamma; \tag{2.286}$$

the additional deflection w^1 is

$$w^1 = -\frac{qR_1^4\delta_1}{16D(1+\nu)(1+\nu+\delta_1)}\cdot\frac{r^2}{R_1^2}. \tag{2.287}$$

Adding the basic deflection (2.285) to the additional deflection (2.287), we obtain the total deflection for the reinforced plate:

$$w = \frac{qR_1^4}{8D}\left[\frac{1}{8}\cdot\frac{r^4}{R_1^4} - \frac{r^2}{R_1^2}\ln\frac{r}{R_1} + \frac{3+\nu+\delta_1}{4(1+\nu+\delta_1)}\cdot\frac{r^2}{R_1^2}\right]. \tag{2.288}$$

Figure 15 shows the maximum deflection of the plate edge $r = R_1$ as a function of δ_1, which characterizes the rigidity of the reinforcing ring. The calculations were performed for $\nu = 0.3$.

It is seen from this figure that the reinforcing ring has a considerable influence on the rigidity of the plate. A relatively thin ring made of the same material as the plate, for which $\delta_1 = 2.7$, reduces the maximum deflections of the plate by 34%.

4) Circular plate with concentric rib, subjected to a load distributed along a circle (Figure 16). The basic deflection for the plate without rib is

$$\text{for } 0 \leqslant r \leqslant \varrho,\quad w_1^0 = \frac{PR_1^2}{8\pi D}\left\{\frac{\left(1-\frac{\varrho^2}{R_1^2}\right)}{2(1+\nu)}\left[(3+\nu)+(\nu-1)\frac{r^2}{R_1^2}\right] - \left(\frac{\varrho^2}{R_1^2}+\frac{r^2}{R_1^2}\right)\ln\frac{R_1}{\varrho}\right\}, \tag{2.289}$$

$$\text{for } \varrho \leqslant r \leqslant R_1,\quad w_0^0 = \frac{PR_1^2}{8\pi D}\left\{\frac{1}{2(1+\nu)}\left[(3+\nu)-(1-\nu)\frac{\varrho^2}{R_1^2}\right]\left(1-\frac{r^2}{R_1^2}\right) - \left(\frac{\varrho^2}{R_1^2}+\frac{r^2}{R_1^2}\right)\ln\frac{R_1}{r}\right\}, \tag{2.290}$$

where P is the total load, distributed uniformly along the circle $r = \varrho$.

The additional deflection w^1 is found by substituting (2.289) in (2.272), with $C^* = 0$:

$$w^1 = -K_0\mu_1(r, k), \tag{2.291}$$

where

$$K_0 = \frac{PR_1^2 k\lambda_1(\delta_1, k)}{8\pi D}\left[\frac{1-\nu}{1+\nu}\left(1-\frac{\varrho^2}{R_1^2}\right)+2\ln\frac{R_1}{\varrho}\right]. \tag{2.292}$$

To obtain the total deflection w we add w^1 (2.291) to the basic deflection w^0, given by (2.289) and (2.290) for the different domains of the plate, and find:

for $0 \leqslant r \leqslant R < \varrho$

$$w = w_1^0 - K_0\left[\frac{2(1-\nu)}{1+\nu}\left(1-\frac{r^2}{R_1^2}\right)+2\left(1-\frac{r^2}{k^2R_1^2}-2\ln k\right)\right], \tag{2.293}$$

for $R \leqslant r < \varrho$

$$w = w_1^0 - K_0\left[\frac{2(1-\nu)}{1+\nu}\left(1-\frac{r^2}{R_1^2}\right)-4\ln\frac{r}{R_1}\right], \tag{2.294}$$

for $R < \varrho < r < R_1$

$$w = w_0^0 - K_0\left[\frac{2(1-\nu)}{1+\nu}\left(1-\frac{r^2}{R_1^2}\right)-4\ln\frac{r}{R_1}\right]. \tag{2.295}$$

If $\varrho < R$ (Figure 17), we determine the additional deflection w^1 by substituting (2.290) in (2.272), with $C^* = 0$:

$$w^1 = -K_1\mu_1(r, k), \tag{2.296}$$

where

$$K_1 = \frac{PR_1^2\lambda_1(\delta_1, k)\,k}{8\pi D}\left[\left(\frac{3+\nu}{1+\nu}-\frac{1-\nu}{1+\nu}\cdot\frac{\varrho^2}{R_1^2}\right)-\frac{1}{k^2}\left(\frac{\varrho^2}{R_1^2}+k^2\right)-2\ln k\right]. \tag{2.297}$$

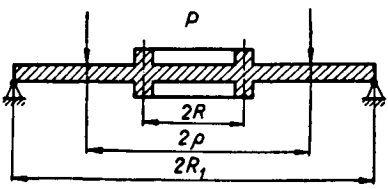


FIGURE 16.

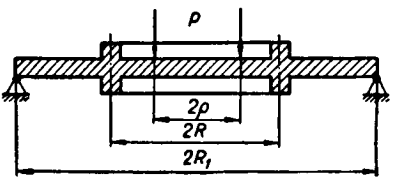


FIGURE 17.

The total deflections of the plate are in this case:

for $0 \leqslant r < \varrho \leqslant R$

$$w = w_1^0 - K_1\left[\frac{2(1-\nu)}{1+\nu}\left(1-\frac{r^2}{R_1^2}\right)+2\left(1-\frac{r^2}{k^2R_1^2}-2\ln k\right)\right], \tag{2.298}$$

for $\varrho < r < R$

$$w = w_0^0 - K_1\left[\frac{2(1-\nu)}{1+\nu}\left(1-\frac{r^2}{R_1^2}\right)+2\left(1-\frac{r^2}{k^2R_1^2}-2\ln k\right)\right],$$

for $\varrho \leqslant R \leqslant r \leqslant R_1$

$$w = w_0^0 - K_1 \left[\frac{2(1-\nu)}{1+\nu} \left(1 - \frac{r^2}{R_1^2} \right) - 4 \ln \frac{r}{R_1} \right]. \tag{2.300}$$

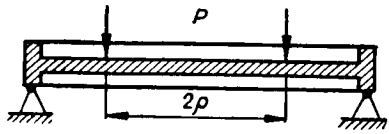


FIGURE 18.

We can, from (2.293) to (2.295) and (2.298) to (2.300), determine the deflections for some particular cases. Thus, for $R=R_1$, i.e., $k=1$, we find from (2.298) and (2.299) the deflections of a plate, supported on its reinforcing rib and subjected to a load distributed along a circle (Figure 18):

For $0 \leqslant r \leqslant \varrho$

$$w = \frac{PR_1^2}{8\pi D} \left\{ \left(1 - \frac{\varrho^2}{R_1^2} \right) \times \right.$$

$$\left. \times \frac{1}{2(1+\nu+\delta_1)} \left[(3+\nu+\delta_1) + (\delta_1+\nu-1) \frac{r^2}{R_1^2} \right] - \frac{\varrho^2+r^2}{R_1^2} \ln \frac{R_1}{\varrho} \right\} \tag{2.301}$$

for $\varrho \leqslant r \leqslant R$

$$w = \frac{PR_1^2}{8\pi D} \left\{ \frac{1}{2(1+\nu+\delta_1)} \left[(3+\nu+\delta_1) + (\delta_1+\nu-1) \frac{\varrho^2}{R_1^2} \right] \left(1 - \frac{r^2}{R_1^2} \right) - \right.$$

$$\left. - \frac{\varrho^2+r^2}{R_1^2} \ln \frac{R_1}{r} \right\}. \tag{2.302}$$

5) Circular plate loaded by the normal pressure $q=\frac{P}{\pi\varrho^2}$ acting on the region $r \leqslant \varrho$.

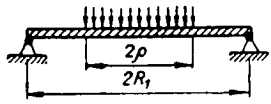
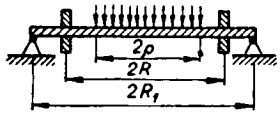
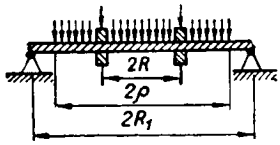
The basic deflection w^0 of a nonreinforced hinged circular plate ($r \leqslant R_1$) is in this case (/ 76/, p. 266):

$$\left. \begin{aligned} w_1^0 &= \frac{P}{16\pi D} \left\{ \left[\frac{2R_1^2}{1+\nu} + \frac{(\nu-1)\varrho^2}{2(1+\nu)} \right] \left(1 - \frac{r^2}{R_1^2} \right) - (\varrho^2 + 2r^2) \ln \frac{R_1}{\varrho} + \right. \right. \\ &\qquad \left. + \frac{r^4}{4\varrho^2} + R_1^2 - \frac{5}{4}\varrho^2 \right\} \qquad (r \leqslant \varrho) \\ w_2^0 &= \frac{P}{16\pi D} \left[(\varrho^2 + 2r^2) \ln \frac{r}{R_1} + \frac{1}{1+\nu} (R_1^2 - r^2) \left(\frac{\nu-1}{2} \cdot \frac{\varrho^2}{R_1^2} + 3 + \nu \right) \right] \\ &\qquad (r \geqslant \varrho) \end{aligned} \right\}. \tag{2.303}$$

Hence, on Γ, i.e., for $r = R$:

$$\left.\begin{aligned} \gamma_1 &= R_1\left(\frac{dw_1^0}{dr}\right)_\Gamma = \frac{PR_1^2}{16\pi D}\left[\frac{2}{1+\nu}\left(\frac{1-\nu}{2}\cdot\frac{\varrho^2}{R_1^2}-2\right)\frac{R}{R_1}-\right.\\ &\qquad\left.-\frac{4R}{R_1}\ln\frac{R_1}{\varrho}+\frac{R^3}{\varrho^2 R_1}\right]\\ \gamma_2 &= R_1\left(\frac{dw_2^0}{dr}\right)_\Gamma = \frac{PR_1^2}{16\pi D}\left[4\frac{R}{R_1}\ln\frac{R}{R_1}+\frac{\varrho^2}{RR_1}+\right.\\ &\qquad\left.+\frac{2}{1+\nu}\cdot\frac{R}{R_1}\left(\frac{1-\nu}{2}\cdot\frac{\varrho^2}{R_1^2}-2\right)\right] \end{aligned}\right\} \qquad (2.304)$$

TABLE 3

Load scheme	Deflections
	$w = w_1^0 \quad (r < \varrho)$ $w = w_2^0 \quad (r \geqslant \varrho)$
	$w = w_1^0 + \mu_{11}(r, k)\lambda_1(\delta_1, k)\gamma_2 \quad (r \leqslant \varrho)$ $w = w_2^0 + \mu_{11}(r, k)\lambda_1(\delta_1, k)\gamma_2 \quad (\varrho \leqslant r \leqslant R)$ $w = w_2^0 + \mu_{10}(r_1)\lambda_1(\delta_1, k)\gamma_2 \quad (r \geqslant R)$
	$w = w_1^0 + \mu_{11}(r, k)\lambda_1(\delta_1, k)\gamma_1 \quad r \leqslant R$ $w = w_1^0 + \mu_{10}(r)\lambda_1(\delta_1, k)\gamma_1 \quad R \leqslant r \leqslant \varrho$ $w = w_2^0 + \mu_{10}(r)\lambda_1(\delta_1, k)\gamma_1 \quad r \geqslant \varrho$

The deflections of a circular plate reinforced by a concentric rib, calculated by (2.303), (2.304), (2.273), and (2.274), are given in Table 3. The coefficients μ_{11}, μ_{10} and λ_1 are given in Tables 1 and 2.

§ 13. SECOND BASIC PROBLEM OF CIRCULAR PLATE WITH CONCENTRIC STIFFENING RIB. EXAMPLE

Let us solve the second basic problem, when $w_0(s)$ and $\frac{\partial w_0}{\partial n}$ are given on the contour $L(r = R_1)$. Boundary condition (2.143) must then be replaced by (2.160); this is equivalent to writing $\varkappa = -1$ in the results of § 11.

This substitution yields:

$$\left.\begin{aligned} &\alpha_0 = 0, \qquad \alpha_1 = -\frac{k^2\delta_1}{2+(1-k^2)\delta_1}\,b_{-1}\\ &\alpha_0' = \frac{4k^2(\delta_1+\delta_2)\,\bar{b}_{-2}}{4+(1-k^4)(\delta_1+\delta_2)}, \qquad -\bar{a}_0 = \frac{2(\delta_1-\delta_2)\,b_{-2}}{4+(1-k^4)(\delta_1+\delta_2)} \end{aligned}\right| \qquad (2.305)$$

$$
\left.\begin{aligned}
a_1 &= \frac{\delta_1 b_{-1}}{2+(1-k^2)\delta_1}, \qquad a_2 = \frac{2(\delta_1+\delta_2)\,\bar{b}_{-2}}{4+(1-k^4)(\delta_1+\delta_2)} \\
a_{-n} &= \frac{2}{\Delta_n'}\left\{\left[(n+2)(1-k^2)\,k^{2n+2}\delta_1\delta_2+\frac{2}{n}(\delta_1-\delta_2)\right]b_{-(n+2)}-\right. \\
&\quad \left.-\left[(1-k^{2n+4})\,\delta_1\delta_2+\frac{2}{n+2}(\delta_1+\delta_2)\right]\bar{b}_n\right\} \\
a_{n+2} &= \frac{2}{\Delta_n'}\left\{\left[\frac{2}{n}(\delta_1+\delta_2)-k^{2n}(1-k^2)\;(n+2)\,n\delta_1\delta_2+\right.\right. \\
&\quad \left.+(1-k^{2n})\,\delta_1\delta_2\right]2\bar{b}_{-(n+2)}-\left[nk^{2n+2}(1-k^2)\,\delta_1\delta_2+\right. \\
&\quad \left.\left.+\frac{2}{n+2}(\delta_1-\delta_2)\right]b_n\right\}
\end{aligned}\right\}, \tag{2.305}
$$

where

$$
\begin{aligned}
\Delta_n' = {} & \frac{16}{n(n+2)}+2\delta_1\left[\frac{2(n+1)}{n(n+2)}-\frac{k^{2n}}{n+2}-nk^{2n}\left(\frac{k^2}{n}+1-k^2\right)^2\right]+ \\
& +2\delta_2\left[\frac{2(n+1)}{n(n+2)}-\frac{k^{2n}}{n+2}-nk^{2n}\left(\frac{k^2}{n}-1+k^2\right)^2\right]+ \\
& +\delta_1\delta_2\{(1+k^{4n+4})-k^{2n}[1+k^4+(n+2)n(1-k^2)^2]\}.
\end{aligned} \tag{2.306}
$$

The coefficients (2.305) define completely the functions $\varphi^0(z)$, $\psi^0(z)$ ((2.253)), $\varphi_0(z)$, $\psi_0(z)$ ((2.254)), and $\varphi_1(z)$, $\psi_1(z)$ ((2.242)). In order to determine the additional deflections w_k^1 of the plate by (2.142), we must calculate the functions

$$
\chi_k(z) = \int \psi_k(z)\,dz + C_k^* \qquad (k=0.1), \tag{2.307}
$$

where C_k^* are real constants, determined from the following conditions:

$$
\left.\begin{aligned}
w_0^1 &= 0 \text{ for } r = R_1 \\
w_1^1 &= w_0^1 \text{ for } r = R
\end{aligned}\right\}. \tag{2.308}
$$

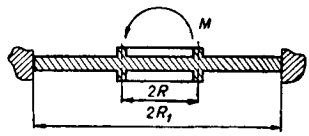


FIGURE 19.

Example. Consider the bending of a circular plate, due to the moment M (Figure 19) created by transverse forces distributed along the contour Γ $(r=R)$ according to the law

$$
p = \frac{M}{\pi R^2}\cos\theta. \tag{2.309}
$$

In the absence of a stiffening rib, the deflection of the plate is

$$\left.\begin{aligned} w_0^0 &= \frac{MR_1}{8\pi D}\left[\frac{a_1'}{R_1}r + a_3'\frac{r^3}{R_1^3} + 2\frac{r}{R_1}\ln\frac{r}{R_1} + a_{-1}'\frac{R_1}{r}\right]\cos\theta \\ w_1^0 &= \frac{MR_1}{8\pi D}\left[b_1'\frac{r}{R_1} + b_3'\frac{r^3}{R_1^3}\right]\cos\theta \end{aligned}\right\}, \tag{2.310}$$

where

$$\left.\begin{aligned} b_3' &= \frac{(1-k^2)^2}{2k^2}, \qquad a_{-1}' = \frac{k^2}{2}, \qquad a_1' = 1-k^2 \\ b_1' &= 1-k^2+2\ln k, \qquad a_3' = \frac{k^2-2}{2} \end{aligned}\right\}. \tag{2.311}$$

For the given case we find [see (1.74)]:

$$\left(\frac{\partial w_k^0}{\partial t}\right)_\Gamma = \frac{1}{2\sigma}\left(\frac{\partial w_k^0}{\partial r} - i\frac{\partial w_k^0}{\partial\theta}\cdot\frac{1}{R}\right)_\Gamma = \frac{M}{32\pi D}[2b_1' + 4b_3'k^2 + 2k^2 b_3'\sigma^{-2}]. \tag{2.312}$$

Substituting (2.311) in (2.312), we obtain the expansion coefficients of (2.247):

$$\left.\begin{aligned} b_0 &= -\frac{M}{32\pi D}[(1-k^2)(2-k^2)+2\ln k] \\ b_{-2} &= -\frac{M}{64\pi D}(1-k^2)^2, \qquad b_n = 0 \quad (n = \pm 1, \quad 2, \pm 3, \pm 4, \ldots) \end{aligned}\right\}. \tag{2.313}$$

Hence, (2.305) becomes

$$\left.\begin{aligned} a_0 = a_1 = a_1 &= 0, \qquad a_n = 0, \qquad (n = -1, -2, \pm 3, \pm 4, \ldots) \\ a_0' &= -\frac{M}{16\pi D}\cdot\frac{k^2(1-k^2)^2(\delta_1+\delta_2)}{4+(1-k^4)(\delta_1+\delta_2)} \\ a_0 &= -\frac{M}{32\pi D}\cdot\frac{(1-k^2)^2(\delta_1-\delta_2)}{4+(1-k^4)(\delta_1+\delta_2)} \\ a_2 &= -\frac{M}{32\pi D}\cdot\frac{(1-k^2)^2(\delta_1+\delta_2)}{4+(1-k^4)(\delta_1+\delta_2)} \end{aligned}\right\}. \tag{2.314}$$

Substitution of (2.314) in (2.254), (2.242), and (2.307) yields:

$$\left.\begin{aligned} \varphi_0(z) &= -k^4 a_2\frac{z^2}{R^2}, \qquad \chi_0(z) = a_0' z - a_2\frac{R^2}{z} + C_0^* \\ \varphi_1(z) &= a_0 + a_2(1-k^4)\frac{z^2}{R^2} \\ \chi_1(z) &= (a_0' - a_0 - 2a_2)z + C_1^* \end{aligned}\right\}. \tag{2.315}$$

Substituting (2.315) in (2.142) and taking (2.308) into account, we obtain the deflections additional to (2.310), due to the stiffening rib:

$$\left.\begin{aligned} w_0^1 &= \frac{MR_1(\delta_1+\delta_2)(1-k^2)^2k^2}{16\pi D\Delta}\left(\frac{r^3}{R_1^3}-2\frac{r}{R_1}+\frac{R_1}{r}\right)\cos\theta \\ w_1^1 &= \frac{MR_1(\delta_1+\delta_2)(1-k^2)}{16\pi Dk^2\Delta}\left[2k^2\frac{r}{R_1}-(1+k^2)\frac{r^3}{R_1^3}\right]\cos\theta \end{aligned}\right\}, \tag{2.316}$$

where

$$\Delta = 4+(1-k^4)(\delta_1+\delta_2). \tag{2.317}$$

The solution of the problem is found by adding (2.316) to (2.310). Letting $(\delta_1+\delta_2)\to\infty$, we obtain, in particular, the solution w_0 given in /43, 213/ for the annular part (domain S_0) of a plate having a central absolutely rigid core.

In domain S_1 the deflections have in this case the form $w_1 = Cr\cos\theta$, i. e., the central disk is rotated as a rigid body.

TABLE 4

k	$\delta_1+\delta_2$					
	0	1	2	5	10	∞
0.5	0.1543	0.1403	0.1309	0.1143	0.1029	0.0809
0.6	0.0768	0.0694	0.0642	0.0550	0.0482	0.0350
0.7	0.0318	0.0288	0.0265	0.0224	0.0192	0.0125
0.8	0.0093	0.0085	0.0079	0.0067	0.0056	0.0032

The angle of rotation of the plane of the elastic stiffening rib is

$$\varphi^* = -\left(\frac{w_1}{r\cos\theta}\right)_\Gamma = -\left(\frac{w_1^0+w_1'}{r\cos\theta}\right) = C^*\frac{M}{Eh^3}, \tag{2.318}$$

where

$$C^* = -\frac{3(1-\nu)}{4\pi}\left[\frac{\delta_1+\delta_2}{\Delta}(1-k^2)^4+(1-k^2)(3-k^2)+4\ln k\right]. \tag{2.319}$$

The coefficient C^* is given in Table 4 for some values of k and $(\delta_1+\delta_2)$ with $\nu = 0.3$.

The data of Table 4 show that the rigidity of the plate at a given load can be considerably increased by means of a stiffening rib.

§ 14. AXISYMMETRICAL BENDING OF A RIGIDLY CLAMPED CIRCULAR PLATE WITH STIFFENING RIB

Consider a circular plate, rigidly clamped at the edge and reinforced by a concentric stiffening rib (Figure 20). The problem of the axisymmetrical

bending of such a plate is solved with the aid of (2.305) (similarly as in § 12) by assuming that all coefficients b_n in (2.247) vanish except for b_{-1}, given by (2.269). Hence,

$$\left.\begin{aligned} a_1 &= -\frac{\delta_1}{4[2+(1-k^2)\delta_1]}\left(\frac{dw^0}{dr}\right)_\Gamma \\ \alpha_1 &= \frac{k^2\delta_1}{4[2+(1-k^2)\delta_1}\left(\frac{dw^0}{dr}\right)_\Gamma \end{aligned}\right\}. \tag{2.320}$$

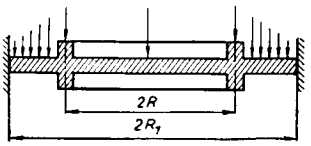


FIGURE 20.

All the remaining coefficients a_n and α_0' vanish. The functions (2.242) and (2.254) become, respectively,

$$\left.\begin{aligned} \varphi_1(z) &= -\frac{(1-k^2)\delta_1}{4[2+(1-k^2)\delta_1]}\left(\frac{dw^0}{dr}\right)_\Gamma\cdot\frac{z}{R}, \qquad \psi_1(z)=0 \\ \varphi_0(z) &= \frac{k^2\delta_1}{4[2+(1-k^2)\delta_1}\left(\frac{dw^0}{dr}\right)_\Gamma\cdot\frac{z}{R} \\ \psi_0(z) &= -\frac{\delta_1}{2[2+(1-k^2)\delta_1]}\left(\frac{dw^0}{dr}\right)_\Gamma\cdot\frac{R}{z} \end{aligned}\right\}. \tag{2.321}$$

We determine the functions $\chi_k(z)$ from (2.307) and the additional deflection, due to the stiffening rib, from (2.271). By satisfying (2.308) we obtain finally:

$$w^1 = \mu_2(r,\ k)\lambda_2(\delta_1,\ k)\,R_1\left(\frac{dw^0}{dr}\right)_\Gamma, \tag{2.322}$$

where $\mu_2(r,\ k)$ acquires, respectively, the values

$$\left.\begin{aligned} \mu_{20} &= -\frac{1}{2}\left(1-\frac{r^2}{R_1^2}\right)-\ln\frac{r}{R_1} \text{ for } R<r<R_1 \\ \mu_{21} &= -\frac{1-k^2}{2k^2}\,\frac{r^2}{R_1^2}-\ln k \text{ for } 0<r<R \end{aligned}\right\}, \tag{2.323}$$

while

$$\lambda_2 = \frac{k\delta_1}{2+(1-k^2)\delta_1}. \tag{2.324}$$

Formulas (2.323) to (2.324) show that the coefficients μ_2 and λ_2, which depend on the rigidity and the location of the rib, are equal for all cases

of axisymmetrical loading of a circular plate. They can be calculated once for all for different values of the parameters δ_1, k, $\frac{r}{R_1}$. The factor $R_1\left(\frac{dw^0}{dr}\right)_\Gamma$ depends on the type of load, but serves only as scale factor for the additional deflections w^1 of the plate

Obviously, for $k=0$ or $k=1$, we have $\left(\frac{dw^0}{dr}\right)_\Gamma = 0$ and therefore $w^1 = 0$.

As in § 12, we obtain simple expressions for the additional bending moments in the plate:

$$\left.\begin{aligned} M_{r1}^1 = M_{\theta 1}^1 &= \frac{\lambda_2 D(1-k^2)(1+\nu)}{R_1 k^2}\left(\frac{dw^0}{dr}\right)_\Gamma \text{ for } r \leqslant R \\ M_{r0}^1 &= -\frac{\lambda_2 D}{R_1}\left(\frac{dw^0}{dr}\right)_\Gamma\left[(1+\nu,+\frac{R_1^2}{r^2}(1-\nu)\right] \\ M_{\theta 0}^1 &= -\frac{\lambda_2 D}{R_1}\left(\frac{dw^0}{dr}\right)_\Gamma\left[(1+\nu)-\frac{R_1^2}{r^2}(1-\nu)\right] \text{ for } R \leqslant r \leqslant R_1 \end{aligned}\right\}. \qquad (2.325)$$

The coefficients μ_2 and λ_2 in (2.322) are always positive. The additional deflection w^1 and the moments M_{r1}^1, $M_{\theta 1}^1$ (2.325) have therefore the same sign as $\left(\frac{dw^0}{dr}\right)_\Gamma$, while the moment M_{r0}^1 has the opposite sign.

Consider the following examples.

1) Rigidly clamped plate subjected to a central force P (Figure 21). We have

$$\left.\begin{aligned} w^0 &= \frac{PR_1^2}{16\pi D}\left[\left(1-\frac{r^2}{R_1^2}\right)+2\frac{r^2}{R_1^2}\ln\frac{r}{R_1}\right] \\ w^1 &= \mu_2\lambda_2\frac{4PR_1^2 k}{16\pi D}\ln k \end{aligned}\right\}.$$

The total deflection is for $r \leqslant R$

$$w = \frac{PR_1^2}{16\pi D}\left[\left(1-\frac{r^2}{R_1^2}\right)+2\frac{r^2}{R_1^2}\ln\frac{r}{R_1}-\frac{4k^2\delta_1\ln k}{2+(1-k^2)\delta_1}\left(\frac{1-k^2}{2k^2}\cdot\frac{r^2}{R_1^2}+\ln k\right)\right], \qquad (2.326)$$

for $r \geqslant R$

$$w = \frac{PR_1^2}{16\pi D}\left\{1-\frac{r^2}{R_1^2}+2\frac{r^2}{R_1^2}\ln\frac{r}{R_1}-\frac{4k^2\delta_1\ln k}{2+(1-k^2)\delta_1}\left[\ln\frac{r}{R_1}+\frac{1}{2}\left(1-\frac{r^2}{R_1^2}\right)\right]\right\}. \qquad (2.327)$$

2) Rigidly clamped plate subjected to a uniformly distributed load q = = const. (Figure 22,a). In this case

$$w^0 = \frac{qR_1^4}{64D}\left(1-\frac{r^2}{R_1^2}\right)^2,$$

$$w_1 = -\frac{qR_1^4}{16D}k(1-k^2)\mu_2\lambda_2.$$

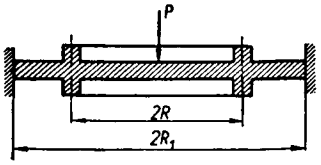


FIGURE 21.

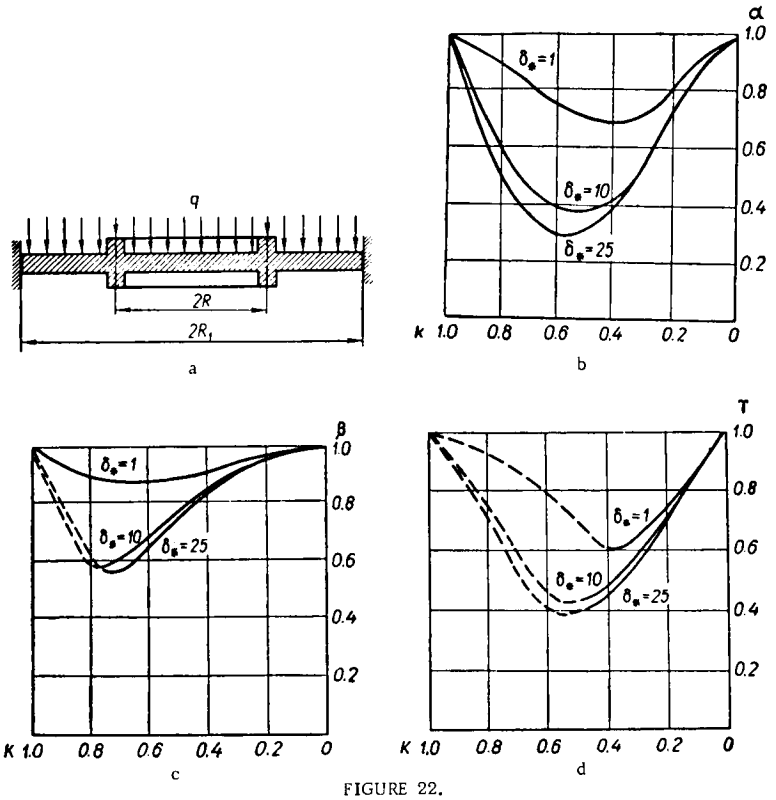


FIGURE 22.

Adding these two expressions, we obtain the total deflection of the reinforced plate:

for $r \leqslant R$

$$w = \frac{qR_1^4}{64D}\left[\left(1-\frac{r^2}{R_1^2}\right)^2 + \frac{4k^2\delta_1(1-k^2)}{2+(1-k^2)\delta_1}\left(\frac{1-k^2}{2k^2}\cdot\frac{r^2}{R_1^2}+\ln k\right)\right], \qquad (2.328)$$

for $r \geqslant R$

$$v = \frac{qR_1^4}{64D}\left\{\left(1-\frac{r^2}{R_1^2}\right)^2 + \frac{4k^2\delta_1(1-k^2)}{2+(1-k^2)\delta_1}\left[\ln\frac{r}{R_1}+\frac{1}{2}\left(1-\frac{r^2}{R_1^2}\right)\right]\right\}. \qquad (2.329)$$

The given case of loading was studied in /210/ from which Figure 22,b-d has been reproduced. Figure 22,b shows the relative decrease of the maximum deflections,

$$\alpha = \frac{w_{max}}{w^0_{max}} = 1 + \frac{4k^2\delta_1(1-k^2)}{2+(1-k^2)\delta_1} \ln k$$

as a function of the rib-location parameter $k = \frac{R}{R_1}$, for three values of the relative rigidity $\delta_* = k\delta_1 = \frac{A}{R_1 D}$. Each value of δ_* corresponds to an optimum value of k, i. e., a position of the rib at which the deflections of the plate are minimum.

Figures 22,c and d show, as functions of the parameters k and δ_*, respectively, the absolute values of the ratios of the maximum normal stresses $(\sigma_r)_{max}$ and $(\sigma_\theta)_{max}$ in a reinforced plate, to the maximum stresses $(\sigma^0_r)_{max}$ and $(\sigma^0_\theta)_{max}$ in a nonreinforced plate of the same dimensions, under the same load. These ratios are

$$\beta = \left| \frac{(\sigma_r)_{max}}{(\sigma^0_r)_{max}} \right| = \left| \frac{(\sigma_r)_{max}}{\frac{3}{4} \cdot \frac{qR_1^2}{h^2}} \right|,$$

$$\gamma = \left| \frac{(\sigma_\theta)_{max}}{(\sigma^0_\theta)_{max}} \right| = \left| \frac{(\sigma_\theta)_{max}}{\frac{3(1+\nu)}{8} \cdot \frac{qR_1^2}{h^2}} \right|.$$

Figure 22,c shows $(\sigma_r)_{max}$ either for $r = R_1$ (full line) or for $r = R$, when approaching the rib from the inside (broken line). The values $(\sigma_\theta)_{max}$ in Figure 22,d are attained either outside the rib (full line), or at $r = 0$ (broken line).

3) Circular plate loaded uniformly along circle $r = \varrho$.

The basic deflection w^0 of the nonreinforced plate with rigidly clamped edge is, respectively:

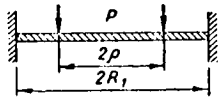
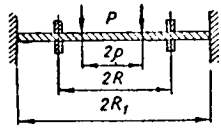
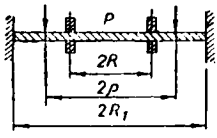
$$\left.\begin{aligned} w^0_1 &= \frac{P}{8\pi D}\left[\frac{1}{2}\left(1+\frac{\varrho^2}{R_1^2}\right)(R_1^2 - r^2) - (r^2+\varrho^2)\ln\frac{R_1}{\varrho} + (r^2-\varrho^2)\right] \quad (r \leqslant \varrho) \\ w^0_2 &= \frac{P}{8\pi D}\left[\frac{1}{2}\left(1+\frac{\varrho^2}{R_1^2}\right)(R_1^2 - r^2) - (r^2+\varrho^2)\ln\frac{R_1}{r}\right] \quad (r > \varrho). \end{aligned}\right\} \tag{2.330}$$

On Γ, i. e., for $r = R$,

$$\left.\begin{aligned} \gamma_3 &= R_1\left(\frac{dw^0_1}{dr}\right)_\Gamma = \frac{PR_1R}{8\pi D}\left[2\left(1-\ln\frac{R_1}{\varrho}\right) - \left(1+\frac{\varrho^2}{R_1^2}\right)\right] \\ \gamma_4 &= R_1\left(\frac{dw^0_2}{dr}\right)_\Gamma = \frac{PR_1R}{8\pi D}\left[\left(1+\frac{\varrho^2}{R^2}\right) - 2\ln\frac{R_1}{R} - \left(1+\frac{\varrho^2}{R_1^2}\right)\right] \end{aligned}\right\}. \tag{2.331}$$

The deflections of a circular plate, rigidly clamped along the edge and reinforced by a concentric thin rib, as obtained from (2.330), (2.331), and (2.322) to (2.324), are given in Table 5.

TABLE 5.

Load scheme	Deflections
	$w=w_1^0 \quad r<\varrho$ $w=w_2^0 \quad r>\varrho$
	$w=w_1^0+\mu_{21}(r, k)\lambda_2(\delta_1, k)Y_4, \quad (r<\varrho)$ $w=w_2^0+\mu_{21}(r, k)\lambda_2(\delta_1, k)Y_4, \quad (\varrho<r\leqslant R)$ $w=w_2^0+\mu_{20}(r)\lambda_2(\delta_1, k)Y_4, \quad (r>R)$
	$w=w_1^0+\mu_{21}(r, k)\lambda_2(\delta_1, k)Y_3, \quad (r\leqslant R)$ $w=w_1^0+\mu_{20}(r)\lambda_2(\delta_1, k)Y_3, \quad (R<r\leqslant\varrho)$ $w=w_2^0+\mu_{20}(r)\lambda_2(\delta_1, k)Y_3, \quad (r>\varrho)$

§ 15. FIRST BASIC PROBLEM FOR AN ELLIPTIC PLATE WITH CENTRAL CIRCULAR STIFFENING RIB. SECOND BASIC PROBLEM. EXAMPLE

Consider an arbitrarily loaded elliptic plate with central circular rib of radius R (Figure 23). We determine the additional stresses, due to the stiffening rib, by representing the unknown single-valued function $g(t)$ defined by (2.151), on $\Gamma(r = R)$ in the form (2.240):

$$g(t)=\sum_{-\infty}^{+\infty} a_n \frac{t^n}{R^n}, \tag{2.332}$$

where a_n are unknown complex constants.

By satisfying the boundary conditions (2.148) to (2.150) on Γ, we find the functions $\Phi^0(z)$, $\Psi^0(z)$ and $\Phi_i(z)$, $\Psi_i(z)\,(i=0,1)$ in the form (2.251), (2.254), (2.242), respectively.

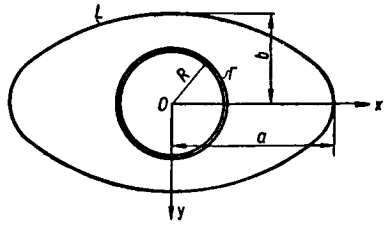


FIGURE 23.

In order to substitute in (2.147) $\varphi_0(z)$ and $\psi_0(z)$ given by (2.254), we obtain from (2.258), (2.261), (2.264):

$$\varphi_0(z) = \sum_{n=1}^{\infty}\left(\beta_n \frac{z^n}{R^n} + \gamma_n \frac{R^n}{z^n}\right), \quad \psi_0(z) = \beta_0' + \sum_{n=1}^{\infty}\left(\beta_n' \frac{z^n}{R^n} + \gamma_n' \frac{R^n}{z^n}\right), \tag{2.333}$$

where

$$\left.\begin{aligned}
&\beta_0' = \alpha_0', \quad \gamma_n = -a_{-n}, \quad \gamma_n' = \bar{a}_n + (2-n)a_{2-n}; \ \beta_n' = \bar{a}_{-n}\left[1 + \right.\\
&\left. + \frac{2}{n}\left(\frac{1}{\delta_1} + \frac{1}{\delta_2}\right) - 2\left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right)\right] - 2(n+2)\bar{b}_{-(n+2)} + a_{n+2} \times\\
&\times\left[(n+2) - \frac{2}{n}\left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right) + 2\left(\frac{1}{\delta_2} + \frac{1}{\delta_1}\right)\right] + 2b_n;\\
&\beta_1 = b_{-1} - a_1\left(1 + \frac{2}{\delta_1}\right) + iJma_1; \ (n = 1, 2, 3, \ldots)\\
&\beta_n = 2\bar{b}_{-n} - a_n\left[1 + \frac{2}{n}\left(\frac{1}{\delta_2} + \frac{1}{\delta_1}\right)\right] + \bar{a}_{-n+2}\left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right)\frac{2}{n}\\
&(n = 2, 3, 4, \ldots)\\
&\delta_1 = \frac{A}{RD}, \quad \delta_2 = \frac{C}{RD}
\end{aligned}\right\} \tag{2.334}$$

On the external contour L of the elliptic plate, whose semiaxes are a and b, we have

$$\tau = R_1\left(\sigma + \frac{m}{\sigma}\right), \quad \bar{\tau} = R_1\left(\frac{1}{\sigma} + m\sigma\right), \tag{2.355}$$

where

$$\sigma = e^{i\theta}, \quad R_1 = \frac{a+b}{2}, \quad m = \frac{a-b}{a+b}. \tag{2.336}$$

Inserting (2.333) and (2.174) into (2.147), we obtain on L:

$$\sum_{1}^{\infty}\left[-\varkappa\bar{\beta}_n\frac{\bar{\tau}^n}{R^n} - \varkappa\bar{\gamma}_n\frac{R^n}{\bar{\tau}^n} + \bar{\tau}n\beta_n\frac{\tau^{n-1}}{R^n} - \bar{\tau}n\gamma_n\frac{R^n}{\tau^{n+1}} + \beta_n'\frac{\tau^n}{R^n} + \gamma_n'\frac{R^n}{\tau^n}\right] + \beta_0' = 0 \text{ on } L. \tag{2.337}$$

With the aid of (2.335), we calculate

$$\left.\begin{aligned}
&\left(\frac{\tau}{R}\right)^n = \sum_{k=-n}^{n} g(k, n)\sigma^{-k}, \quad \left(\frac{\tau}{R}\right)^{-n} = \sum_{k=n}^{\infty} g_1(k-1, n-1)\sigma^{-k}\\
&\left(\frac{\bar{\tau}}{R}\right)^n = \sum_{k=-n}^{n} g(k, n)\sigma^{k}, \quad \left(\frac{\bar{\tau}}{R}\right)^{-n} = \sum_{k=n}^{\infty} g_1(k-1, n-1)\sigma^{k}
\end{aligned}\right\}, \tag{2.338}$$

where

$$g(k, n) = \frac{n|\eta^n m^{\frac{k+n}{2}}}{\frac{k+n}{2}|\frac{n-k}{2}|}, \quad \eta = \frac{R_1}{R} \tag{2.339}$$

$$g_1(k,\ n) = (-1)^{\frac{k-n}{2}} m^{\frac{k-n}{2}} \eta^{-(n+1)} \frac{\left(\frac{k+n}{2}\right)!}{\left(\frac{k-n}{2}\right)!n!} \qquad \text{(2.339) cont'd}$$

The coefficients $g(n,\ k)$ and $g_1(n,\ k)$ are meaningful only when the numbers $n+k$ and $n-k$ are even.

In order to obtain an approximate solution of the problem, we retain only a finite number of terms in (2.332) and (2.247):

$$g(t) = \sum_{n=-\gamma}^{r} a_n \frac{t^n}{R^n}, \qquad (2.340)$$

$$R(t) = -2 \sum_{n=-(\gamma+2)}^{r} b_n \sigma^n - \frac{1}{2i} C_{g1}. \qquad (2.341)$$

We then have, according to (2.334), instead of (2.333):

$$\left.\begin{aligned} \varphi_0(z) &= \sum_{n=1}^{\gamma+2} \beta_n \frac{z^n}{R^n} + \sum_{n=1}^{r} \gamma_n \frac{R^n}{z^n} \\ \psi_0(z) &= \sum_{n=0}^{r} \beta'_n \frac{z^n}{R^n} + \sum_{n=1}^{\gamma+2} \gamma'_n \frac{R^n}{z^n} \end{aligned}\right\}, \qquad (2.342)$$

The boundary condition (2.337) becomes

$$\sum_{n=1}^{r} \left\{ -\varkappa \bar{\beta}_n \left(\frac{\bar{\tau}}{R}\right)^n - \varkappa \bar{\gamma}_n \left(\frac{\bar{\tau}}{R}\right)^{-n} + \left[\frac{\bar{\tau}}{R}(n+1)\beta_{n+1} + \beta'_n\right]\left(\frac{\tau}{R}\right)^n + \right.$$
$$\left. + \left[\gamma'_n - \frac{\bar{\tau}}{R}(n-1)\gamma_{n-1}\right]\left(\frac{\tau}{R}\right)^{-n} \right\} + \left(\beta'_0 + \beta_1 \frac{\bar{\tau}}{R}\right) = 0. \qquad (2.343)$$

Substituting (2.338) in (2.343) and rearranging, we obtain

$$\sum_{n=1}^{r} \left\{ -\sum_{k=-n}^{n} \varkappa \bar{\beta}_n g(k,\ n)\sigma^k - \sum_{k=n}^{\infty} \varkappa \bar{\gamma}_n g_1(k-1,\ n-1)\sigma^k + \right.$$
$$+ \sum_{k=1-n}^{n+1} \eta \beta_{n+1}(n+1) g(k-1,\ n)\sigma^{-k} +$$
$$+ \sum_{k=-1-n}^{n-1} m(n+1)\beta_{n+1} g(k+1,\ n)\eta\sigma^{-k} + \sum_{k=-n}^{n} \beta'_n g(k,\ n)\sigma^{-k} +$$
$$+ \sum_{k=n}^{\infty} \gamma'_n g_1(k-1,\ n-1)\sigma^{-k} - \sum_{k=n+1}^{\infty} (n-1)\gamma_{n-1}\eta g_1(k-2,\ n-1)\sigma^{-k} -$$
$$\left. - \sum_{k=n-1}^{\infty} m(n-1)\gamma_{n-1}\eta g_1(k,\ n-1)\sigma^{-k} \right\} + \beta'_0 + \beta_1\eta(\sigma^{-1} + m\sigma) = 0. \qquad (2.344)$$

Equating to zero the coefficient of the free term in (2.344), we obtain

$$-\sum_{n=2,4,6,\ldots}^{r}\varkappa\bar{\beta}_n g(0,n)+\sum_{n=1,3,5,\ldots}^{r}\eta(n+1)[g(-1,n)+ mg(1,n)]\beta_{n+1}+\sum_{n=2,4,6,\ldots}^{r} g(0,n)\beta_n'+\beta_0'=0. \tag{2.345}$$

Equating the coefficients of σ^k, we obtain for $k=\pm 1$:

$$\left.\begin{aligned}
&-\sum_{n=1,3,5,\ldots}^{r}\varkappa\bar{\beta}_n g(-1,n)+\sum_{n=2,4,6,\ldots}^{r}\eta(n+1)[g(0,n)+\\
&+mg(2,n)]\beta_{n+1}+\beta_1\eta+\sum_{n=1,3,5,\ldots}^{r}[g(1,n)\beta_n'+\\
&+g_1(0,n-1)\gamma_n']-\sum_{n=2,4,6,\ldots}^{r}m\eta g_1(1,n-1)\gamma_{n-1}(n-1)=0,\\
&-\sum_{n=1,3,5,\ldots}^{r}\varkappa g(1,n)\bar{\beta}_n+\sum_{n=2,4,6,\ldots}^{r}(n+1)\eta[g(-2,n)+\\
&+mg(0,n)]\beta_{n+1}+\sum_{n=1,3,5,\ldots}^{r}g(-1,n)\beta_n'-\\
&-\sum_{n=1,3,5,\ldots}^{r}\varkappa g_1(0,n-1)\bar{\gamma}_n+m\eta\beta_1=0
\end{aligned}\right\}. \tag{2.346}$$

From (2.339) follow the equalities

$$\left.\begin{aligned} g(-k,n)=0,\quad & g(k,n)=0 \text{ for } k>n\\ & g_1(k,n)=0 \text{ for } k<n\end{aligned}\right\}. \tag{2.347}$$

Equating to zero the coefficients of $\sigma^k (k=\pm 2, \pm 3, \ldots, \pm r)$ in (2.344), we obtain

$$\left.\begin{aligned}
&\sum_{n=1}^{r}\{-\bar{\beta}_n\varkappa g(-k,n)+[g(k-1,n-1)+mg(k+1,n-1)]n\eta\beta_n+\\
&+g(k,n)\beta_n'+g_1(k-1,n-1)\gamma_n'-[g_1(k-2,n)+\\
&+mg_1(k,n)]n\eta\gamma_n\}=0\\
&\sum_{n=1}^{r}\{-\bar{\beta}_n\varkappa g(k,n)-\varkappa g_1(k-1,n-1)\bar{\gamma}_n+g(-k,n)\beta_n'+\\
&+[g(-k-1,n-1)+mg(-k+1,n-1)]n\eta\beta_n\}=0
\end{aligned}\right\}. \tag{2.348}$$

We have thus obtained the $(1+2r)$ linear algebraic equations (2.345), (2.346), and (2.348). If we add to these the single-valuedness condition (2.246) for the function $P(t)$, and an equal number of equations obtained for the conjugate magnitudes, we obtain $(4r+4)$ equations altogether.

These equations contain $(2r+3)$ coefficients $a_{-r},\ldots,a_0, a_1,\ldots,a_r$, $\operatorname{Im}\alpha_1$, α_0', and $(2r+1)$ conjugate coefficients $\bar{a}_{-r},\ldots,\bar{a}_0,\bar{a}_2,\ldots,\bar{a}_r,\bar{a}_0$ (a_1 and $\operatorname{Im}\alpha_1$ are real), i.e., $(4r+4)$ unknowns altogether.

The number of equations is thus equal to the number of unknowns. By solving system (2.345), (2.346), (2.348) we obtain all coefficients of

the function $g(t)$, defined by (2.340), through which the coefficients of the functions $\varphi_0(z)$, $\psi_0(z)$ ((2.342)) and $\varphi_1(z)$, $\psi_1(z)$ ((2.242)) are also expressed.

An approximate solution of the problem is thus obtained.

To solve the second basic problem, in which $w_0(s)$ and $\frac{\partial w^0}{\partial n}$ are given on the contour L, the boundary condition (2.147) must be replaced by (2.160); this is equivalent to setting $\varkappa = -1$ in (2.337), (2.343) to (2.348).

Example. An elliptic plate, rigidly clamped along the contour, is reinforced by a central circular stiffening rib of radius R and is subjected to a uniformly distributed load q.

The basic deflection of a nonreinforced plate is (/136/, p. 279)

$$w^0 = w^*\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2, \tag{2.349}$$

where

$$w^* = \frac{q}{D}\left(\frac{24}{a^4} + \frac{24}{b^4} + \frac{16}{a^2b^2}\right)^{-1}, \tag{2.350}$$

a and b are the semiaxes of the plate.

We find

$$\left(\frac{\partial w^0}{\partial t}\right)_\Gamma = \frac{1}{2}\left(\frac{\partial w^0}{\partial x} - i\frac{\partial w^0}{\partial y}\right)_\Gamma = -2\sum_{-3}^{3} b_n \sigma_n, \tag{2.351}$$

where

$$\left.\begin{aligned} b_1 &= \frac{Rw^*}{2}\left[\frac{1}{a^2} - \frac{3R^2}{4a^4} - \frac{1}{b^2} + \frac{3R^2}{4b^4}\right] \\ b_{-1} &= \frac{Rw^*}{2}\left[\frac{1}{a^2} - \frac{3R^2}{4a^4} - \frac{R^2}{2a^2b^2} - \frac{3R^2}{4b^4} + \frac{1}{b^2}\right] \\ b_3 &= -\frac{R^3w^*}{8a^4b^4}(b^2 - a^2)^2, \quad b_{-3} = \frac{R^3w^*}{8a^4b^4}(a^4 - b^4) \end{aligned}\right\}. \tag{2.352}$$

If we retain only six terms ($r = 3$) in the expansion (2.340) of the unknown function $g(t)$, we obtain from (2.345) to (2.348) and (2.246) a system of eight equations, from which we find the real coefficients a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3 and β_0'. (These equations are not given here since they are very involved.) In addition $\operatorname{Im} a_1 = 0$ in the case considered.

§16. ANOTHER FORM OF THE BOUNDARY CONDITIONS

The boundary conditions (2.148) to (2.150) at the junction between plate and stiffening rib can also be written in a different form, which is convenient for solving many problems (cf. Chapter VI).

Let us write (2.42) in the form

$$\frac{L_{\tau k}}{C_k} - i\frac{L_{nk}}{A_k} = i\frac{\partial}{\partial s}\left[\frac{1}{i}\left(\frac{\partial w_k}{\partial n} + i\frac{\partial w_k}{\partial s}\right)\right]. \tag{2.353}$$

Separating the real and imaginary parts and using (2.16), we obtain

$$L_{\tau k} = C_k\left(\frac{1}{\varrho}\frac{\partial w_k}{\partial s} + \frac{\partial^2 w_k}{\partial n \partial s}\right), \quad L_{nk} = A_k\left(\frac{1}{\varrho}\frac{\partial w_k}{\partial n} - \frac{\partial^2 w_k}{\partial s^2}\right). \tag{2.354}$$

Substituting (2.354) in (2.19) and then in the first equation (2.11), we find

$$\left.\begin{aligned} m_k(s) &= \frac{A_k}{\varrho_k}\left(\frac{1}{\varrho_k}\cdot\frac{\partial w_k}{\partial n} - \frac{\partial^2 w_k}{\partial s^2}\right) - \frac{\partial}{\partial s}\left[C_k\left(\frac{1}{\varrho_k}\cdot\frac{\partial w_k}{\partial s} + \frac{\partial^2 w}{\partial n \partial s}\right)\right] \\ p_k(s) &= \frac{\partial}{\partial s}\left\{\frac{\partial}{\partial s}\left[A_k\left(\frac{1}{\varrho_k}\cdot\frac{\partial w_k}{\partial n} - \frac{\partial^2 w_k}{\partial s^2}\right)\right] + \right. \\ &\quad \left. + \frac{C_k}{\varrho_k}\left(\frac{\partial^2 w_k}{\partial n \partial s} + \frac{1}{\varrho_k}\cdot\frac{\partial w_k}{\partial s}\right)\right\} \end{aligned}\right\}. \tag{2.355}$$

We have, in addition, by (2.4) and (2.5):

$$w_0 = w_k, \quad \frac{\partial w_0}{\partial n} = \frac{\partial w_k}{\partial n} \text{ on } \gamma_k. \tag{2.356}$$

The left-hand sides of (2.355) are also expressed through the functions w_0 and w_k which determine the deflections of the plate at the rib γ_k. In fact, according to (2.3), the functions $m_k(s)$ and $p_k(s)$ represent the jump (across γ_k) in the bending moments $m_{0k}(s)$, $m_{kk}(s)$, and in the generalized shearing forces $p_{0k}(s)$, $p_{kk}(s)$; these again are expressed through the deflections w_0 and w_k by (1.57), in which w must be replaced by w_0 and w_k, respectively.

Formulas (2.355) and (2.356) represent the conditions at the junction between the plate and the stiffening rib or, which is the same, between the domains S_0 and S_k of the plate. These conditions contain no additional unknown functions or constants and are expressed only through the unknown deflections w_0 and w_k and their derivatives, and also through the known rigidities A_k and C_k of the rib. It is assumed that the flexural rigidities of the various domains $S_k (k = 1, 2, \ldots, l)$ differ, being, respectively, equal to D_k.

Thus, when γ_k is a circle of radius $\varrho_k = R$, the boundary conditions (2.355) to (2.356), represented in a polar system of coordinates (r, θ) with origin at the center of the circle $\gamma_k \left(\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}\right)$, are for $r = R$

$$\left.\begin{aligned} &(\delta_1 + \nu_k - \lambda_k\nu_0)\frac{\partial w_k}{\partial r} + R\frac{\partial^2 w_k}{\partial r^2} - \lambda_k R\frac{\partial^2 w_0}{\partial r^2} - \delta_2\frac{\partial^3 w_k}{\partial r \partial \theta^2} + \\ &+ \frac{1}{R}(\delta_1 + \delta_2 + \nu_k - \lambda_k\nu_0)\frac{\partial^2 w_k}{\partial \theta^2} = -m_0(\theta)\frac{R}{D_k}, \\ &(1 - \lambda_k)\frac{\partial w_k}{\partial r} - R\frac{\partial^2 w_k}{\partial r^2} + R\lambda_k\frac{\partial^2 w_0}{\partial r^2} - \\ &- [\delta_2 + \nu_k - 3 + \lambda_k(3 - \nu_0)]\frac{1}{R}\cdot\frac{\partial^2 w_k}{\partial \theta^2} - R^2\frac{\partial^3 w_k}{\partial r^3} + \end{aligned}\right\} \tag{2.357}$$

$$\left.\begin{aligned} &+\lambda_k R^3\frac{\partial^3 w_0}{\partial r^3}+[\delta_1+\delta_2-2+\nu_k+\lambda_k(2-\nu_0)]\frac{\partial^3 w_k}{\partial r\partial\theta^2}+\\ &+\frac{\delta_1}{R}\frac{\partial^4 w_k}{\partial\theta^4}=-p_0(\theta)\frac{R^3}{D_k},\\ &w_k=w_0,\quad \frac{\partial w_k}{\partial r}=\frac{\partial w_0}{\partial r}, \end{aligned}\right\} \tag{2.357}$$

where

$$\delta_1=\frac{A_k}{RD_k},\quad \delta_2=\frac{C_k}{RD_k},\quad \lambda_k=\frac{D_0}{D_k},$$

$p_0(\theta)$ and $m_0(\theta)$ are given external transverse forces and moments applied to the rib; ν_k is Poisson's ratio for the material in domain S_k.

If the flexural rigidity of the plate is uniform, i.e., $\lambda=1$ and $\nu_0=\nu_k$, the boundary conditions are simplified:

$$\left.\begin{aligned} &\delta_1\frac{\partial w_k}{\partial r}+R\frac{\partial^2}{\partial r^2}(w_k-w_0)+\frac{\delta_1+\delta_2}{R}\cdot\frac{\partial^2 w_k}{\partial\theta^2}-\delta_2\frac{\partial^3 w_k}{\partial r\partial\theta^2}=-m_0(\theta)\frac{R}{D_k}\\ &-R\frac{\partial^2 w_k}{\partial r^2}+R\frac{\partial^2 w_0}{\partial r^2}-\frac{\delta_2}{r}\cdot\frac{\partial^2 w_k}{\partial\theta^2}-R^2\frac{\partial^3}{\partial r^3}(w_k-w_0)+\\ &+(\delta_1+\delta_2)\frac{\partial^3 w_k}{\partial r\partial\theta^2}+\frac{\delta_1}{R}\cdot\frac{\partial^4 w_k}{\partial\theta^4}=-p_0(\theta)\frac{R^2}{D_k}\\ &w_k=w_0,\quad \frac{\partial w_k}{\partial r}=\frac{\partial w_0}{\partial r} \end{aligned}\right\}. \tag{2.358}$$

If the rib is located at the edge of a circular plate, i.e., $\nu_0=D_0=0$, or $\lambda_k=0$, (2.357) becomes

$$\left.\begin{aligned} &(\delta_1+\nu_k)\frac{\partial w_k}{\partial r}+R\frac{\partial^2 w_k}{\partial r^2}+\frac{\delta_1+\delta_2+\nu_k}{R}\cdot\frac{\partial^2 w_k}{\partial\theta^2}-\\ &-\delta_2\frac{\partial^3 w_k}{\partial r\partial\theta^2}=-m_0(\theta)\frac{R}{D_k}\\ &\frac{\partial w_k}{\partial r}-R\frac{\partial^2 w_k}{\partial r^2}-\frac{1}{R}(\delta_2+\nu_k-3)\frac{\partial^2 w_k}{\partial\theta^2}-R^2\frac{\partial^3 w_k}{\partial r^3}+\\ &+(\delta_1+\delta_2+\nu_k-2)\frac{\partial^3 w_k}{\partial r\partial\theta^2}+\frac{\delta_1}{R}\cdot\frac{\partial^4 w_k}{\partial\theta^4}=-p_0(\theta)\frac{R^2}{D_k} \end{aligned}\right\}. \tag{2.359}$$

When the stiffening rib is located at the edge of a circular hole in the plate, $D_k=0$ and (2.357) becomes

$$\left.\begin{aligned} &(\delta_1'-\nu_0)\frac{\partial w_0}{\partial r}-R\frac{\partial^2 w_0}{\partial r^2}+\frac{1}{R}(\delta_1'+\delta_2'-\nu_0)\frac{\partial^2 w_0}{\partial\theta^2}-\\ &-\delta_2'\frac{\partial^3 w_0}{\partial r\partial\theta^2}=m_0(\theta)\frac{R}{D_0}\\ &-\frac{\partial w_0}{\partial r}+R\frac{\partial^2 w_0}{\partial r^2}-\frac{1}{R}(\delta_2+3-\nu_0)\frac{\partial^2 w_0}{\partial\theta^2}+R^2\frac{\partial^3 w_0}{\partial r^3}+\\ &+(\delta_1'+\delta_2'+2-\nu_0)\frac{\partial^3 w_0}{\partial r\partial\theta^2}+\frac{\delta_1'}{R}\cdot\frac{\partial^4 w_0}{\partial\theta^4}=p_0(\theta)\frac{R^2}{D_0} \end{aligned}\right\} \tag{2.360}$$

where

$$\delta_1' = \frac{A_k}{RD_0}, \quad \delta_2' = \frac{C_k}{RD_0}. \tag{2.361}$$

The positive directions of $m_0(\theta)$ and $p_0(\theta)$ coincide here with the positive directions of M_r and Q_r on the inner contour of the plate.

If the reinforcing rib forms a support along which the deflections vanish, i.e., $w_k = \frac{\partial w_k}{\partial s} = \frac{\partial^2 w_k}{\partial s^2} = \frac{\partial^4 w_k}{\partial s^4} = 0$, the second boundary condition (2.355) is replaced by $w_k = 0$, while the first becomes

$$m_k(s) = \frac{A_k}{\varrho_k^2} \cdot \frac{\partial w_k}{\partial n} - C_k \frac{\partial^2 w_k}{\partial n \partial s^2}. \tag{2.362}$$

It is interesting to note that the support moment and the angle of rotation of the plate edge are directly proportional only in the case of axisymmetrical loading of the circular plate.

Conditions (2.356) remain unaltered.

In this case the second equation (2.355) serves for determining the unknown support reactions along γ_k; if these reactions are denoted by $N_k(s)$, we obtain

$$N_k(s) = p_k(s) - \frac{\partial^2}{\partial s^2}\left(\frac{A_k}{\varrho_k} \cdot \frac{\partial w_k}{\partial n}\right) - \frac{C_k}{\varrho_k} \cdot \frac{\partial^2 w_k}{\partial n \partial s}. \tag{2.363}$$

For a circular supporting rib (2.359) and (2.360) become:

1) Outer supporting rib:

$$(\delta_1 + \nu)\frac{\partial w}{\partial r} + R\frac{\partial^2 w}{\partial r^2} - \delta_2 \frac{\partial^2 w}{\partial r \partial \theta^2} = -m_0(\theta) \cdot \frac{R}{D_k}, \quad w = 0 \tag{2.364}$$

2) Inner supporting rib:

$$(\delta_1' - \nu)\frac{\partial w}{\partial r} - R\frac{\partial^2 w}{\partial r^2} - \delta_2' \frac{\partial^2 w}{\partial r \partial \theta^2} = m_0(\theta)\frac{R}{D_0}, \quad w = 0, \tag{2.365}$$

where the subscripts of w and ν have been omitted.

In the case of axisymmetrical bending of a circular or annular plate with reinforced edge we have $w = w(r)$. Boundary conditions (2.359) and (2.360) are then considerably simplified.

3) For an outer reinforced plate edge $r = R$,

$$\left.\begin{aligned} (\delta_1 + \nu)\frac{dw}{dr} + R\frac{d^2 w}{dr^2} &= -m_0 \frac{R}{D_k} \\ \frac{dw}{dr} - R\frac{d^2 w}{dr^2} - R^2 \frac{d^3 w}{dr^3} &= -p_0 \frac{R^2}{D_k} \end{aligned}\right\}. \tag{2.366}$$

4) For an inner reinforced edge $r = R$ of an annular plate

$$\left.\begin{aligned} (\delta_1' - \nu)\frac{dw}{dr} - R\frac{d^2w}{dr^2} &= m_0\frac{R}{D_0} \\ -\frac{dw}{dr} + R\frac{d^2w}{dr^2} + R^2\frac{d^3w}{dr^3} &= p_0\frac{R^2}{D_0} \end{aligned}\right\}. \tag{2.367}$$

5) For a plate reinforced by a concentric circular rib of radius R [not at the edge], we obtain from (2.357):

$$\left.\begin{aligned} (\nu_1 - \lambda_1\nu_0 + \delta_1)\frac{dw_1}{dr} + R\frac{d^2}{dr^2}(w_1 - \lambda_1 w_0) &= -m_0\frac{R}{D_1} \\ w_1 = w_0,\quad \frac{dw_1}{dr} &= \frac{dw_0}{dr} \\ (1-\lambda_1)\frac{dw_1}{dr} - R\frac{d^2w_1}{dr^2} + R\lambda_1\frac{d^2w_0}{dr^2} - & \\ -R^2\frac{d^3}{dr^3}(w_1 - \lambda_1 w_0) &= -p_0\frac{R^2}{D_1} \end{aligned}\right\}, \tag{2.368}$$

where subscript "1" corresponds to $r \leqslant R$, and subscript "0" to $r \geqslant R$; $\lambda_1 = D_0/D_1$.

If the stiffening rib forms a support, the second condition (2.366) and the last condition (2.368) are replaced by $w = 0$.

The boundary conditions introduced in this section will be used frequently in the following chapters.

§ 17. DYNAMICAL BOUNDARY CONDITIONS

In dynamical problems of motion of thin plates reinforced by ribs, the boundary conditions (2.355) and (2.356) remain in force, but to the external forces $p_k(s)$ and moments $m_k(s)$ acting on the rib we must add, in accordance with d'Alembert's principle, the inertia loads

$$-\mu F w_k'', \quad -\mu I_\varepsilon \theta_{\tau k}'', \tag{2.369}$$

where μ is the density of the rib material; F is the cross-sectional area of the rib, $\theta_{\tau k} = \frac{\partial w}{\partial n}$ is the angle of rotation of the rib section about the tangent τ, μI_ε is the moment of inertia of unit length of the rib referred to the axis passing through the center of bending. The double primes denote the second partial derivative with respect to time t.

The dynamical boundary conditions at the junction between the thin plate and an elastic rib located on it are thus:

$$\left.\begin{aligned} m_k(s) - \mu I_e \theta_{\tau k}^{\cdot} &= \frac{A_k}{\varrho_k}\left(\frac{1}{\varrho_k}\cdot\frac{\partial w_k}{\partial n} - \frac{\partial^2 w_k}{\partial s^2}\right) - \frac{\partial}{\partial s}\left[C_k\left(\frac{1}{\varrho_k}\frac{\partial w_k}{\partial s} + \frac{\partial^2 w_k}{\partial n \partial s}\right)\right] \\ p_k(s) - \mu F w_k^{\cdot} &= \frac{\partial}{\partial s}\left\{\frac{\partial}{\partial s}\left[A_k\left(\frac{1}{\varrho_k}\frac{\partial w_k}{\partial n} - \frac{\partial^2 w_k}{\partial s^2}\right)\right] + \right. \\ &\left. + \frac{C_k}{\varrho_k}\left(\frac{\partial^2 w_k}{\partial n \partial s} + \frac{1}{\varrho_k}\frac{\partial w_k}{\partial s}\right)\right\}, \quad w_k = w_0, \quad \frac{\partial w_k}{\partial n} = \frac{\partial w_0}{\partial n} \end{aligned}\right\}. \tag{2.370}$$

In the case of free vibrations of the plate we represent the deflection w_k in polar coordinates in the form

$$w_k(r, \theta, t) = W_k(r, \theta)\cos(pt + \varphi_0). \tag{2.371}$$

When the axial line of the rib forms a circle of radius $\varrho_k = R$, (2.370) becomes, by analogy with (2.357), for $r = R$

$$\left.\begin{aligned} &(\delta_1 + \nu_k - \lambda_k\nu_0 - \alpha_1 x_1^4)\frac{\partial W_k}{\partial r} + R\frac{\partial^2}{\partial r^2}(W_k - \lambda_k W_0) + \\ &+ \frac{1}{R}(\delta_1 + \delta_2 + \nu_k - \lambda_k\nu_0)\frac{\partial^2}{\partial\theta^2}W_k - \delta_2\frac{\partial^3}{\partial r\partial\theta^2}W_k = 0 \\ &- \alpha_2 x_1^4 W_k + R(1-\lambda_k)\frac{\partial}{\partial r}W_k - R^2\frac{\partial^2}{\partial r^2}(W_k - \lambda_k W_0) + \\ &+ \delta_1\frac{\partial^4}{\partial\theta^4}W_k - [\delta_2 - 3 + \nu_k + \lambda_k(3-\nu_0)]\frac{\partial^2}{\partial\theta^2}W_k - \\ &- R^3\frac{\partial^3}{\partial r^3}(W_k - \lambda_k W_0) + R[\delta_1 + \delta_2 - 2 + \nu_k + \lambda_k(2-\nu_0)]\times \\ &\times\frac{\partial^3}{\partial r\partial\theta^2}W_k = 0 \end{aligned}\right\}, \tag{2.372}$$

$$W_k = W_0, \quad \frac{\partial}{\partial r}W_k = \frac{\partial}{\partial r}W_0, \tag{2.373}$$

where

$$\alpha_1 = \frac{\mu I_e}{\mu_k R^3 h_k}, \quad \alpha_2 = \frac{\mu F}{\mu_k R h_k}, \quad x_1 = k_1 R, \tag{2.374}$$

$$k_1^4 = p^2/c^4, \quad c^4 = E_k h_k^2/12\mu_k(1-\nu_k^2), \quad \lambda_k = \frac{D_0}{D_k},$$

h_k, μ_k are, respectively, thickness and density of the plate in S_k.

Particular cases of conditions (2.372) and (2.373) for circular and annular plates with reinforced edges are easily obtained in the same way as in the preceding section (cf. § 5 of Chapter VI).

In some terms of (2.372) we may replace W_0 by W_k and vice versa, by virtue of (2.373) and the following equalities:

$$\frac{\partial^2 W_0}{\partial\theta^2} = \frac{\partial^2 W_k}{\partial\theta^2}, \quad \frac{\partial^3 W_0}{\partial r\partial\theta^2} = \frac{\partial^3 W_k}{\partial r\partial\theta^2}, \quad \frac{\partial^4 W_0}{\partial\theta^4} = \frac{\partial^4 W_k}{\partial\theta^4} \text{ on } \gamma_k. \tag{2.375}$$

If the stiffening rib forms a support along which the deflections vanish, i. e., $w = \frac{\partial w}{\partial s} = \frac{\partial^2 w}{\partial s^2} = \frac{\partial^4 w}{\partial s^4} = 0$, (2.370) becomes

$$\left.\begin{aligned} m_k(s) - \mu I_\sigma \theta^*_{\tau k} &= \frac{A_k}{\varrho_k^2} \cdot \frac{\partial w_k}{\partial n} - C_k \frac{\partial^3 w_k}{\partial n \partial s^2} \\ w_0 = w_k = 0, \quad \frac{\partial w_0}{\partial n} &= \frac{\partial w_k}{\partial n} \end{aligned}\right\}. \tag{2.376}$$

In the first condition (2.376) we may replace w_k by w_0.

Chapter III

BENDING OF THIN PLATES WITH REINFORCED EDGES

§ 1. STATEMENT OF THE PROBLEM. BOUNDARY CONDITIONS. DEGREE OF DETERMINACY OF THE FUNCTIONS

Let the middle plane of an arbitrarily loaded thin plate occupy the interior or exterior of a simple smooth closed contour Γ. The plate is reinforced along Γ by a thin elastic rib of different material. The rib has a constant cross section, one of whose principal axes of inertia lies in the middle plane of the plate (Figure 24).

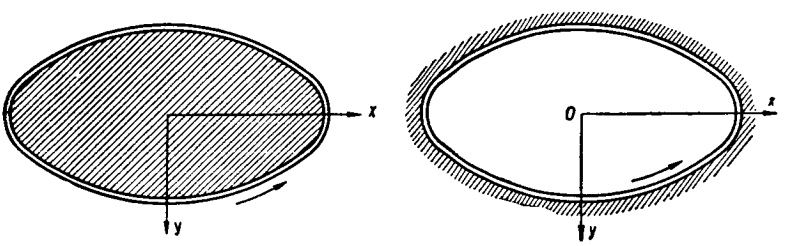


FIGURE 24.

The additional moments and strains of the plate, due to the reinforcing rib, are given by the functions $\varphi(z)$ and $\psi(z)$, which are analytic and single-valued in the plate domain and satisfy on Γ the conditions of juncture between plate and rib.

In § 8 of Chapter II we derived four boundary conditions for the more general case of a stiffening rib located in the middle plane of the plate.

Two boundary conditions suffice in the particular case considered here: the condition that the forces and moments of interaction between the thin rib and the plate be equal, and the condition that the strains in the rib and plate be equal. The first of these conditions is obviously obtained from (2.148), by assuming, e. g., that no load acts on the rib from the side of S_0, i. e., that $\varphi^+(t) = \varphi'^+(t) = \psi^+(t) = 0$. The second condition coincides with (2.150).

We have thus*

$$-\varkappa\overline{\varphi(t)} + t\overline{\varphi'(t)} + \psi(t) = -(\varkappa+1)\overline{g(t)}, \tag{3.1}$$

$$\overline{\varphi(t)} + t\overline{\varphi'(t)} + \psi(t) = P(t) - R(t), \tag{3.2}$$

* If the plate occupies the region outside Γ, the sign of the right-hand side of (3.1) is inverted.

where [cf. (2.151) to (2.153)]:

$$\left.\begin{aligned} g(t) &= \frac{1}{8D}[iC_6 t + \bar{C}_7 - \overline{I(t)}] = \\ &= \frac{1}{8D}\left\{\int_0^s [m(s) + i\int_0^{s_1} p(s_2)\,ds_2]\,\dot{t}\,ds_1 + iC_6 t + \bar{C}_7\right\} \\ P(t) &= \frac{1}{4i}\int_0^s \left\{\left(\frac{1}{C}+\frac{1}{A}\right)[C_4 - iV_b^0(\bar{t}-\bar{t}_0) + I(t)] + \right. \\ &\left. + \left(\frac{1}{C}-\frac{1}{A}\right)[\bar{C}_4 + iV_b^0(t-t_0) + \overline{I(t)}]\,\dot{\bar{t}}^2\right\} ds \\ R(t) &= \frac{\partial w^0}{\partial t} - \frac{1}{2i}C_3 \end{aligned}\right\} \qquad (3.3)$$

Here $I(t)$ is an unknown function depending on the unknown moments and forces with which the plate acts on the rib along Γ, and is given by (2.25); $w^0 = w^0(x, y)$ is the basic deflection, assumed to be known, of the same plate without stiffening rib, under the same load.

In view of the arbitrariness in determining the functions $\varphi(z)$, $\psi(z)$, and $\chi(z) = \int\psi(z)dz$ and of the considerations in § 9 of Chapter II, we conclude that under the specified stresses in a finite plate we can assume [cf. (2.174) and (2.175)] that

$$C_6 = -V_b^0, \quad C_7 = -C_4 - iV_b^0\bar{t}_0, \quad \varphi(z_0) = \chi(z_0) = 0, \qquad (3.4)$$

where z_0 is any point in S_0. We can replace the condition $\varphi(z_0) = 0$ in (3.4) by $\psi(z_0) = 0$.

Conditions (3.4) eliminate completely the indeterminacy in the functions giving the solution of the problem stated.

If the deflections of the plate are specified in addition to the stresses, then, by analogy with (2.180) and (2.181), we can eliminate the indeterminacy in the functions introduced by writing

$$C_6 = -V_b^0, \quad C_7 = -C_4 - iV_b^0\bar{t}_0, \quad \operatorname{Im}[\chi(z_0)] = 0. \qquad (3.5)$$

We can use (3.4) or (3.5) to express the unknown function $P(t)$ in (3.3) through the unknown function $g(t)$ [cf. (2.182)]:

$$P(t) = 2iD\int_0^s \left[\left(\frac{1}{C}+\frac{1}{A}\right)\overline{g(t)} + \left(\frac{1}{C}-\frac{1}{A}\right)\dot{\bar{t}}^2 g(t)\right] ds. \qquad (3.6)$$

The uniqueness theorem is proved in the case considered exactly as in § 5 of Chapter II.

We note only one particular case, which will be useful to us in the next section: if the contour Γ of a nonreinforced plate undergoes only a rigid-body displacement, i. e.,

$$\frac{\partial w^0}{\partial t} = v = \text{const on } \Gamma, \qquad (3.7)$$

the stresses in the plate are not affected by the insertion of an elastic rib. In fact, we find by direct substitution that the boundary conditions (3.1) to (3.2) are satisfied if we assume that the additional state of stress of the plate, due to the stiffening rib, vanishes identically:

$$\left.\begin{array}{c} \varphi(z) = \alpha_0 = \text{const}, \quad \psi(z) = \varkappa\bar{\alpha}_0 = \text{const} \\ g(t) = 0, \quad C_3 = 2i(v + \bar{\alpha}_0 + \varkappa\bar{\alpha}_0) \end{array}\right\}. \tag{3.8}$$

By virtue of the uniqueness of the solution, we therefore conclude that (3.8) solves the problem under condition (3.7). In accordance with (2.91), (2.47), and (2.48), the internal forces and moments in the rib vanish; by (2.18), the angles of rotation θ_τ and θ_n of the rib sections define only a rigid-body displacement:

$$\theta_\tau - i\theta_n = C_3 \dot{t}. \tag{3.9}$$

§ 2. CASE OF FINITE DOMAIN. REDUCTION OF THE PROBLEM TO FREDHOLM'S INTEGRAL EQUATION

The problem stated in the preceding section can be reduced to a Fredholm integral equation of the second kind. We note that the function $g(t)$ in (3.1) is related with the function $P(t)$ by (3.6), and can be easily expressed through $\frac{dP(t)}{dt} = P'(t)$ in the form

$$g(t) = \frac{i}{8D}[(A + C)\dot{t}\overline{P'(t)} + (A - C)\dot{t}^3 P'(t)]. \tag{3.10}$$

Subtracting (3.1) from (3.2) and introducing conjugate magnitudes we obtain

$$\varphi(t) = \frac{1 - \nu}{4}[\overline{P(t)} - \overline{R(t)}] + g(t). \tag{3.11}$$

We also find from (3.1):

$$\psi(t) = \varkappa\overline{\varphi(t)} - \bar{t}\varphi'(t) - (\varkappa + 1)\overline{g(t)}. \tag{3.12}$$

Thus, if the function $P(t)$ and hence also $P'(t)$ are known, it is possible to obtain from (3.11) and (3.12) the boundary values $\varphi(t)$ and $\psi(t)$ of the functions $\varphi(z)$ and $\psi(z)$, holomorphic in the domain within the simple smooth contour Γ. The functions themselves are given by Cauchy's formula /104/:

$$\varphi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(t)\,dt}{t - z}, \tag{3.13}$$

$$\psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varkappa\overline{\varphi(t)} - \bar{t}\varphi'(t) - \overline{g(t)}(\varkappa + 1)}{t - z}\,dt. \tag{3.14}$$

We obtain the boundary values of $\varphi(z)$ and $\psi(z)$ for $z \to t$ from Sokhotskii's formulas. Their substitution in (3.1) yields:

$$-\frac{\varkappa}{2\pi i}\int_{\Gamma}\left(\frac{d\bar{t}}{\bar{t}-\bar{t}_0}+\frac{dt}{t-t_0}\right)\overline{\varphi(t)}+\frac{1}{2\pi i}\int_{\Gamma}\frac{\bar{t}-\bar{t}_0}{t-t_0}\varphi'(t)\,dt=$$

$$=\frac{\varkappa+1}{2}\overline{g(t_0)}-\frac{\varkappa+1}{2\pi i}\int_{\Gamma}\frac{\overline{g(t)}\,dt}{t-t_0}. \tag{3.15}$$

The second integral on the left side of this equation is easily integrated by parts; because of the single-valuedness of $\varphi(t)$ on Γ, we obtain

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\bar{t}-\bar{t}_0}{t-t_0}\varphi'(t)\,dt=-\frac{1}{2\pi i}\int_{\Gamma}\varphi(t)\,d\frac{\bar{t}-\bar{t}_0}{t-t_0}.$$

Introducing conjugate magnitudes we can rewrite (3.15) as follows:

$$-\frac{\varkappa}{2\pi i}\int_{\Gamma}\varphi(t)\left(\frac{dt}{t-t_0}+\frac{d\bar{t}}{\bar{t}-\bar{t}_0}\right)-\frac{1}{2\pi i}\int_{\Gamma}d\frac{t-t_0}{\bar{t}-\bar{t}_0}\overline{\varphi(t)}+$$

$$+\frac{\varkappa+1}{2}\left[g(t_0)+\frac{1}{\pi i}\int_{\Gamma}\frac{g(t)\,d\bar{t}}{\bar{t}-\bar{t}_0}\right]=0. \tag{3.16}$$

Substituting in (3.16) for $\varphi(t)$ and $g(t)$ from (3.11) and (3.10), respectively, we obtain the following singular integral-differential equation:

$$a_1(t_0)P'(t_0)+b_1(t_0)\overline{P'(t_0)}+\frac{1}{\pi i}\int_{\Gamma}[\Gamma_0^*(t_0,t)P(t)+$$

$$+\Gamma_1^*(t_0,t)P'(t)]\frac{dt}{t-t_0}+\frac{1}{\pi i}\int_{\Gamma}[\Omega_0^*(t_0,t)\overline{P(t)}+$$

$$+\Omega_1^*(t_0,t)\overline{P'(t)}]\frac{dt}{t-t_0}=f(t_0), \tag{3.17}$$

where

$$a_1(t_0)=-2iDt_0^3\left(\frac{1}{C}-\frac{1}{A}\right),\quad b_1(t_0)=-2iD\bar{t}_0^3\left(\frac{1}{C}+\frac{1}{A}\right), \tag{3.18}$$

$$\Gamma_0^*(t_0,t)=\frac{(1-\nu)^2D^2(t-t_0)}{AC}\cdot\frac{d}{dt}\cdot\frac{t-t_0}{\bar{t}-\bar{t}_0},$$

$$\Omega_0^*(t_0,t)=\frac{(1-\nu)(3+\nu)D^2}{AC}\left(1+\bar{t}^2\frac{t-t_0}{\bar{t}-\bar{t}_0}\right), \tag{3.19}$$

$$\Gamma_1^*(t_0,t)=-\frac{1-\nu}{4}\overline{b_1(t)}(t-t_0)\frac{d}{dt}\cdot\frac{t-t_0}{\bar{t}-\bar{t}_0}-\frac{3+\nu}{4}a_1(t)+\frac{1-\nu}{4}a_1(t)\bar{t}^2\frac{t-t_0}{\bar{t}-\bar{t}_0}, \tag{3.20}$$

$$\Omega_1^*(t_0,t)=b_1(t)\frac{t-t_0}{\bar{t}-\bar{t}_0}\bar{t}^2-\frac{3+\nu}{4}b_1(t)\left(1+\frac{t-t_0}{\bar{t}-\bar{t}_0}\bar{t}^2\right)-$$

$$-\frac{1-\nu}{4}\overline{a_1(t)}(t-t_0)\frac{d}{dt}\cdot\frac{t-t_0}{\bar{t}-\bar{t}_0},$$

$$f(t_0)=\frac{(1-\nu)^2D^2}{\pi iAC}\int_\Gamma R(t)\,d\frac{t-t_0}{\bar t-\bar t_0}+$$
$$+\frac{(1-\nu)(3+\nu)D^2}{\pi iAC}\int_\Gamma \overline{R(t)}\left(\frac{dt}{t-t_0}+\frac{d\bar t}{\bar t-\bar t_0}\right). \tag{3.21}$$

Substituting in (3.21) for $R(t)$ from (3.3), and remembering that

$$\int_\Gamma d\frac{t-t_0}{\bar t-\bar t_0}=0,\quad \int_\Gamma\left(\frac{dt}{t-t_0}+\frac{d\bar t}{\bar t-\bar t_0}\right)=0,$$

we can rewrite (3.21) so that it no longer contains the unknown constant C_3:

$$f(t_0)=\frac{(1-\nu)^2D^2}{\pi iAC}\int_\Gamma \frac{\partial w^0}{\partial t}\,d\frac{t-t_0}{\bar t-\bar t_0}+$$
$$+\frac{(1-\nu)(3+\nu)D^2}{\pi iAC}\int_\Gamma \frac{\partial w^0}{\partial t}\left(\frac{dt}{t-t_0}+\frac{d\bar t}{\bar t-\bar t_0}\right). \tag{3.22}$$

Since the function $P(t)$ is single-valued on Γ, we find, integrating by parts:

$$\frac{1}{\pi i}\int_\Gamma \Gamma_0^*(t_0,\ t)\frac{P(t)\,dt}{t-t_0}=-\frac{(1-\nu)^2D^2}{\pi iAC}\int_\Gamma \frac{t-t_0}{\bar t-\bar t_0}P'(t)\,dt. \tag{3.23}$$

Similarly,

$$\frac{1}{\pi i}\int_\Gamma \Omega_0^*(t_0,\ t)\overline{P(t)}\frac{dt}{t-t_0}=-\frac{2(1-\nu)(3+\nu)D^2}{\pi iAC}\int_\Gamma \overline{P'(t)}\,\bar t^2\ln|t-t_0|dt. \tag{3.24}$$

Substituting (3.23) and (3.24) in (3.17), we obtain

$$a_1(t_0)P'(t_0)+b_1(t_0)\overline{P'(t_0)}+\frac{1}{\pi i}\int_\Gamma \Gamma(t_0,\ t)P'(t)\frac{dt}{t-t_0}+$$
$$+\frac{1}{\pi i}\int_\Gamma \Omega(t_0,\ t)\overline{P'(t)}\frac{dt}{t-t_0}=f(t_0), \tag{3.25}$$

where

$$\left.\begin{aligned}\Gamma(t_0,\ t)&=\Gamma_1^*(t_0,\ t)-\frac{(1-\nu)^2D^2}{AC}\cdot\frac{(t-t_0)^2}{\bar t-\bar t_0}\\ \Omega(t_0,\ t)&=\Omega_1^*(t_0,\ t)-\frac{2}{AC}(1-\nu)(3+\nu)D^2(t-t_0)\bar t^2\ln|t-t_0|\end{aligned}\right\}. \tag{3.26}$$

We have thus obtained the singular integral equation (3.25) for determining the function $P'(t)$; this equation contains, in addition to the unknown function, also its complex conjugate $\overline{P'(t)}$.

We introduce the new function

$$\mu_0(t) = P'(t)\,\dot{t}^2 \tag{3.27}$$

and rewrite (3.25) in the form

$$\frac{D}{AC}(A-C)\,\mu_0(t_0) + \frac{D}{AC}(A+C)\,\overline{\mu_0(t_0)} + \frac{1}{\pi i}\int_\Gamma K_1(t_0,\ t)\,\mu_0(t)\,\frac{dt}{t-t_0} +$$

$$+ \frac{1}{\pi i}\int_\Gamma K_2(t_0,\ t)\,\overline{\mu_0(t)}\,\frac{dt}{t-t_0} = F(t_0), \tag{3.28}$$

where

$$K_1(t_0, t) = \frac{i}{2}\,\dot{t}_0\,\dot{\bar{t}}^2\Gamma(t_0, t), \quad K_2(t_0,\ t) = \frac{i}{2}\,\dot{t}_0\,\dot{t}^2\,\Omega(t_0,\ t),$$

$$F(t_0) = \frac{i}{2}\,\dot{t}_0 f(t_0). \tag{3.29}$$

Adding to (3.28) the corresponding equation in complex conjugate magnitudes, we obtain the following system:

$$\Lambda_\alpha\mu \equiv \sum_{\beta=1}^{2} A^*_{\alpha\beta}\mu_\beta(t_0) + \frac{1}{\pi i}\int_\Gamma \sum_{\beta=1}^{2} K_{\alpha\beta}(t_0, t)\,\mu_\beta(t)\,\frac{dt}{t-t_0} =$$

$$= f^0_\alpha(t_0) \quad (\alpha = 1,2), \tag{3.30}$$

where

$$\left.\begin{aligned} A^*_{11} = A^*_{22} = A - C, \quad A^*_{12} = A^*_{21} = A + C \\ f^0_1(t_0) = ACF(t_0)/D, \quad f^0_2(t_0) = \overline{f^0_1(t_0)} \end{aligned}\right\}, \tag{3.31}$$

$$\mu_1(t) = \mu_0(t), \quad \mu_2(t) = \overline{\mu_0(t)}, \tag{3.32}$$

$$\left.\begin{aligned} K_{11}(t_0,\ t) = K_1(t_0,\ t)\,AC/D, \quad K_{12}(t_0,\ t) = K_2(t_0,\ t)\,AC/D \\ K_{21}(t_0,\ t) = -\overline{K_{12}(t_0,\ t)}\,\frac{t-t_0}{\bar{t}-\bar{t}_0}\,\dot{\bar{t}}^2 \\ K_{22}(t_0,\ t) = -\overline{K_{11}(t_0,\ t)}\,\frac{t-t_0}{\bar{t}-\bar{t}_0}\,\dot{\bar{t}}^2 \end{aligned}\right\}. \tag{3.33}$$

In matrix notation (3.30) becomes

$$\Lambda\mu \equiv A^*\mu(t_0) + \frac{1}{\pi i}\int_\Gamma K(t_0,\ t)\,\mu(t)\,\frac{dt}{t-t_0} = f^0(t_0), \tag{3.34}$$

where

$$A^* = \|A^*_{\alpha\beta}\|, \quad K(t_0,\ t) = \|K_{\alpha\beta}(t_0,\ t)\|$$

are known matrices on Γ, satisfying everywhere on Γ the Hölder condition (H), (the matrix $K(t_0,\ t)$ by both variables); $f^0(t_0)=(f^0_1,\ f^0_2)$ is a given vector,

satisfying the H condition everywhere on Γ: $\mu(t) = (\mu_1, \mu_2)$ is an unknown vector also satisfying the H condition.

We rewrite (3.34) in the form

$$\Lambda\mu \equiv A^*\mu(t_0) + \frac{B}{\pi i}\int_\Gamma \frac{\mu(t)\,dt}{t-t_0} + \frac{1}{\pi i}\int_\Gamma K_0(t_0, t)\,\mu(t)\,dt = f^0(t_0), \tag{3.35}$$

where

$$K_0(t_0, t) = \frac{K(t_0, t) - B}{t - t_0}, \tag{3.36}$$

$$A^* = \begin{Vmatrix} A-C & A+C \\ A+C & A-C \end{Vmatrix}, \quad B = \frac{1+\nu}{2}\begin{Vmatrix} -(A-C) & -(A+C) \\ A+C & A-C \end{Vmatrix}. \tag{3.37}$$

The determinants of the matrices

$$S = A^* + B, \quad T = A^* - B \tag{3.38}$$

are

$$\det S = \det T = -(1-\nu)(3+\nu)AC \neq 0. \tag{3.39}$$

Equation (3.35) is therefore of the normal type.

The total index of the equation $\Lambda\mu = 0$ is

$$\varkappa_0 = \frac{1}{2\pi i}[\ln\det T - \ln\det S]_\Gamma = 0. \tag{3.40}$$

Consider the characteristic equation corresponding to (3.35):

$$M\mu \equiv A^*\mu(t_0) + \frac{B}{\pi i}\int_\Gamma \frac{\mu(t)\,dt}{t-t_0} = f_*(t). \tag{3.41}$$

This equation reduces /11/ to the Hilbert problem

$$\Phi^+(t_0) = G\Phi^-(t_0) + g_*(t_0), \tag{3.42}$$

where

$$g_*(t_0) = S^{-1}f_*(t_0), \quad G = S^{-1}T. \tag{3.43}$$

The canonical matrix $X(z)$, corresponding to the homogeneous Hilbert problem (obtained from (3.42) for $g_* = 0$), is

$$\Phi^+(t_0) = G\Phi^-(t_0), \tag{3.44}$$

where $G = \|G_{\alpha\beta}\|$. According to (3.43), the elements $G_{\alpha\beta}$ are real nonzero constants. It is easily shown that

$$\det G = -1$$

everywhere on Γ. In this case the canonical system of solutions of the homogeneous Hilbert problem (3.44) is represented by the system of vectors $\overset{k}{\Phi}(z)\,(k=1,2)$ defined by /11/

$$\left.\begin{aligned}\overset{k}{\Phi}(z) &= G\overset{k}{\gamma} \text{ for } z\in S_0 = S^+\\ \overset{k}{\Phi}(z) &= \overset{k}{\gamma} \text{ for } z\in S^-\end{aligned}\right\}. \tag{3.45}$$

In our case $\overset{1}{\gamma}=(1,0)$, $\overset{2}{\gamma}=(0,1)$($S^-$ is the region completing $S_0+\Gamma$ to a complete plane). Hence

$$\overset{1}{\Phi}(z)=(G_{11},\ G_{21}),\quad \overset{2}{\Phi}(z)=(G_{12},\ G_{22}) \text{ for } z\in S^+,$$
$$\overset{1}{\Phi}(z)=(1,0),\quad \overset{2}{\Phi}(z)=(0,1) \text{ for } z\in S^-.$$

The canonical matrices are thus

$$X^+=\begin{Vmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{Vmatrix}=G,\quad X^-=\begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix}. \tag{3.46}$$

The determinants of these matrices differ everywhere from zero.

The general solution, vanishing at infinity, of the nonhomogeneous Hilbert problem (3.42) is

$$\Phi(z)=\frac{X(z)}{2\pi i}\int_\Gamma [X^+(t)]^{-1}g_*(t)\frac{dt}{t-z}=\frac{X(z)G^{-1}}{2\pi i}\int_\Gamma \frac{g_*(t)\,dt}{t-z}. \tag{3.47}$$

Since all the particular indices of problem (3.42) are equal to zero, a solution of (3.41) always exists, and contains no arbitrary constants.

Having obtained the solution (3.47), we can find an expression for the required solution $\mu(t)$ of (3.41) in the form /11/:

$$\mu(t_0)=A_0f_*(t_0)+\frac{B_0Z}{\pi i}\int_\Gamma f_*(t_1)\frac{dt_1}{t_1-t_0}, \tag{3.48}$$

where

$$A_0=\frac{1}{2}[S^{-1}+T^{-1}]=\frac{1}{(1-\nu)(3+\nu)AC}\begin{Vmatrix}-(A-C) & A+C\\ A+C & -(A-C)\end{Vmatrix}, \tag{3.49}$$

$$B_0=\frac{1}{2}[X^+-X^-]=\frac{1}{2}\begin{Vmatrix}G_{11}-1 & G_{12}\\ G_{21} & G_{22}-1\end{Vmatrix}, \tag{3.50}$$

$$Z=[X^-]^{-1}T^{-1}=$$
$$=\frac{1}{2(1-\nu)(3+\nu)AC}\begin{Vmatrix}-(1-\nu)(A-C) & (3+\nu)(A+C)\\ (1-\nu)(A+C) & -(3+\nu)(A-C)\end{Vmatrix}. \tag{3.51}$$

We return now to (3.35) whose general solution is given by (3.48), with

$$f_*(t_0) = f^0(t_0) - \frac{1}{\pi i}\int_\Gamma K_0(t_0, t)\,\mu(t)\,dt. \tag{3.52}$$

Substituting in (3.48) for $f_*(t_0)$ from (3.52), we obtain a Fredholm integral equation of the second kind, equivalent to (3.35):

$$\mu(t_0) + \frac{1}{\pi i}\int_\Gamma K^*(t_0, t)\,\mu(t)\,dt = F_*(t_0), \tag{3.53}$$

where

$$F_*(t_0) = A_0 f^0(t_0) + \frac{B_0 Z}{\pi i}\int_\Gamma \frac{f^0(t_1)\,dt_1}{t_1 - t_0}, \tag{3.54}$$

$$K^*(t_0, t) = \|K^*_{\alpha\beta}\| = A_0 K_0(t_0, t) + \frac{B_0 Z}{\pi i}\int_\Gamma \frac{K_0(t_1, t)\,dt_1}{t_1 - t_0}. \tag{3.55}$$

Fredholm's alternative is applicable to (3.53). Hence, in order to prove that it can be solved for any right-hand side, it suffices to show that the corresponding homogeneous equation has no nontrivial solutions.

Consider the homogeneous equation obtained from (3.53) for $F_*(t_0) = 0$, i.e.,

$$A_0 f^0(t_0) + \frac{B_0 Z}{\pi i}\int_\Gamma \frac{f^0(t_1)\,dt_1}{t_1 - t_0} = 0. \tag{3.56}$$

Using designations (3.49) to (3.51), we can rewrite the last equation in the form

$$f^0(t_0) + \left\|\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}\right\| \cdot \frac{1+\nu}{2\pi i}\int_\Gamma \frac{f^0(t)\,dt}{t - t_0} = 0$$

or

$$f_1^0(t_0) + \frac{1+\nu}{2\pi i}\int_\Gamma \frac{f_1^0(t)\,dt}{t - t_0} = 0, \quad f_2^0(t_0) - \frac{1+\nu}{2\pi i}\int_\Gamma \frac{f_2^0(t)\,dt}{t - t_0} = 0. \tag{3.57}$$

Solving the system of homogeneous singular integral equations (3.57), we obtain

$$f_1^0(t) = f_2^0(t) = 0. \tag{3.58}$$

According to (3.31) and (3.29) we then also have $F(t_0) = 0$ and

$$f(t_0) = 0. \tag{3.59}$$

By virtue of (3.21), this can be written in the form

$$\frac{1}{2\pi i}\int_{\Gamma} R(t)\,d\frac{t-t_0}{\bar{t}-\bar{t}_0}+\frac{3+\nu}{1-\nu}\cdot\frac{1}{2\pi i}\int_{\Gamma}\overline{R(t)}\left(\frac{dt}{t-t_0}+\frac{d\bar{t}}{\bar{t}-\bar{t}_0}\right)=0. \tag{3.60}$$

This singular integral equation of the first kind gives the solution of the first basic problem in the theory of bending of thin plates when no load is applied (cf. (3.16)). This solution is

$$R(t)=\text{const on } \Gamma$$

or, according to (3.3),

$$\frac{\partial w^0}{\partial t}=v=\text{const on } \Gamma. \tag{3.61}$$

Condition (3.61) is thus a consequence of the vanishing of the right-hand side of (3.53).

Let $\mu_*(t)$ be any solution of the homogeneous equation corresponding to (3.53), and let $P'_*(t)$, $g_*(t)$, $\varphi_*(z)$, $\psi_*(z)$ be the corresponding values of the functions $P'(t)$, $g(t)$, $\varphi(z)$, $\psi(z)$ defined respectively by (3.27), (3.10), (3.13), (3.14), if $\mu_0(t)$ is replaced by $\mu_*(t)$ in (3.27).

The functions $\varphi_*(z)$ and $\psi_*(z)$ satisfy the boundary conditions (3.1) and (3.2) under condition (3.56), and thus solve our problem under condition (3.61); but in this case we have, according to (3.8):

$$\varphi_*(z)=\alpha_0=\text{const},\quad \psi_*(z)=\varkappa\bar{\alpha}_0,\quad g_*(t)=0,$$

and by (3.6) and (3.27)

$$P_*(t)=P'_*(t)=\mu_*(t)=0 \text{ on } \Gamma. \tag{3.62}$$

The homogeneous equation corresponding to (3.53) thus has no nontrivial solutions. Hence, (3.53) has always one and only one solution $\mu(t)$.

Substituting this value of $\mu(t)$ in (3.32), we find $\mu_0(t)$ and thence $P'(t)$ from (3.27).

Knowing $P'(t)$, we determine $P(t)$ by integration, except for an arbitrary constant, and also the function $g(t)$ from (3.10). We then obtain $\varphi(t)$ from (3.11), and finally express by (3.13) and (3.14) the functions $\varphi(z)$ and $\psi(z)$ holomorphic in the domain of the plate, which characterize the additional stresses in the plate, due to the stiffening rib. The problem is thus solved.

Equation (3.53) can be solved by the usual approximative methods /3, 48, 69, 79, 132/.

§ 3. SOME RELATIONSHIPS FOR AN UNBOUNDED PLATE WITH CURVED HOLE

Let bending moments $M_x^{(\infty)}=$ const. and $M_y^{(\infty)}=$ const. act at infinity on an unbounded plate having a hole whose contour is Γ. The deflections w of

such a plate satisfy the biharmonic equation, and therefore the sum

$$M_x + M_y = -D(1+\nu)\Delta w \tag{3.63}$$

is a harmonic function. The sum (3.63) satisfies Green's identity (/39/, Chapter 1, § 8):

$$\int_{\Gamma_*} \frac{d(M_x + M_y)}{dn}\, ds + \int_{\Gamma_1} \frac{d}{dn}(M_x + M_y)\, ds = 0, \tag{3.64}$$

where Γ_* is an arbitrary smooth closed contour surrounding Γ, and Γ_1 is a circle of infinite radius (Figure 25); $\frac{d}{dn}$ is the derivative along the normal n to the corresponding contour.

Since in our case $\frac{d}{dn}(M_x + M_y) = 0$ at infinity, (3.64) reduces to

$$\int_{\Gamma_*} \frac{d(M_x + M_y)}{dn}\, ds = 0. \tag{3.65}$$

We introduce polar coordinates (ϱ, θ):

$$z = x + iy = \omega(\zeta) = \omega(\varrho e^{i\theta}),$$

where $\omega(\zeta)$ is a function mapping conformally the exterior of Γ_* onto the exterior of the unit circle $\gamma(\varrho = 1)$ on the plane $\zeta = \varrho e^{i\theta}$. Since the mapping is conformal, we have

$$\frac{dn}{d\varrho} = \frac{ds}{d\theta}. \tag{3.66}$$

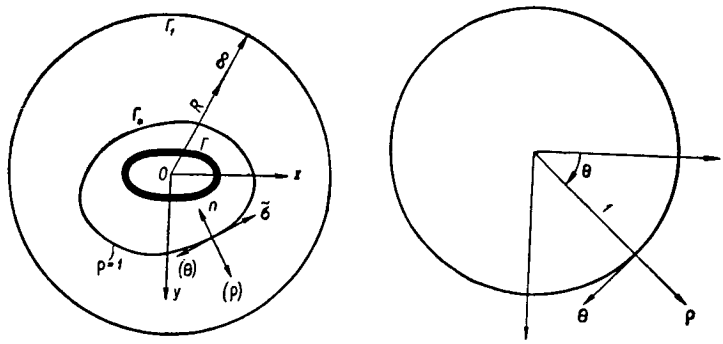


FIGURE 25.

In addition, we obtain from the first invariant of the stress tensor

$$M_x + M_y = M_\varrho + M_\theta = M_n + M_\tau \tag{3.67}$$

on Γ_*. Using (3.66) and (3.67), we rewrite (3.65) in the form

$$\int_{\Gamma_*} \frac{d}{dn}(M_x + M_y)\frac{dn}{d\varrho}\,d\theta = \frac{d}{d\varrho}\int_0^{2\pi}(M_n + M_\tau)\,d\theta = 0,$$

where

$$\int_0^{2\pi}(M_n + M_\tau)\,d\theta = C = \text{const}, \tag{3.68}$$

i. e., (3.68) is constant for all values of ϱ. The constant C is found by evaluating (3.68) for a circle of infinite radius, where

$$M_n + M_\tau = M_x^{(\infty)} + M_y^{(\infty)},$$

whence

$$C = 2\pi(M_x^{(\infty)} + M_y^{(\infty)}). \tag{3.69}$$

Substituting (3.69) in (3.68), we find

$$\frac{1}{2\pi}\int_0^{2\pi}(M_n + M_\tau)\,d\theta = M_x^{(\infty)} + M_y^{(\infty)}. \tag{3.70}$$

This shows that the mean value of the sum of the bending moments $(M_n + M_\tau)_{\Gamma_*}$ on any circle in the plane ζ is equal to the sum of the bending moments at infinity $M_x^{(\infty)} + M_y^{(\infty)}$.

A similar result was obtained by Brock /201/ in the two-dimensional problem of the theory of elasticity.

If the plate is reinforced along Γ_* by a thin rib, we obtain from (3.70)

$$\frac{1}{2\pi}\int_0^{2\pi}(M_n^- + M_\tau^-)\,d\theta = M_x^{(\infty)} + M_y^{(\infty)}. \tag{3.71}$$

In the absence of a stiffening rib we have, according to (3.70), for the same contour Γ_*

$$\frac{1}{2\pi}\int_0^{2\pi}(M_n^0 + M_\tau^0)\,d\theta = M_x^{(\infty)} + M_y^{(\infty)}, \tag{3.72}$$

where M_n^0 M_τ^0 are the bending moments on Γ_*, corresponding to the basic state of stress of the plate without rib.

Equating (3.71) with (3.72), we find

$$\frac{1}{2\pi}\int_0^{2\pi}(M_n^- - M_n^0)\,d\theta = -\frac{1}{2\pi}\int_0^{2\pi}(M_\tau^- - M_\tau^0)\,d\theta, \tag{3.73}$$

i. e., the respective increments averaged over γ, of the bending moments M_n and M_τ on Γ_* (in domain S^-), due to the stiffening rib, are equal in magnitude and opposite in sign.

If we now assume that Γ_* delimits the hole in the plate ($\Gamma_* = \Gamma$), then $M_n^0 = 0$ on Γ_*, and by (3.72),

$$\frac{1}{2\pi}\int_0^{2\pi} M_\tau^0 d\theta = M_x^{(\infty)} + M_y^{(\infty)}, \tag{3.74}$$

i. e., the mean value on γ of the circumferential bending moments is equal to the sum of the bending moments $M_x^{(\infty)}$ and $M_y^{(\infty)}$, acting on the plate at infinity.

Hence, for pure torsion of the plate ($M_x^{(\infty)} = -M_y^{(\infty)}$), we have

$$\frac{1}{2\pi}\int_0^{2\pi} M_\tau d\theta = 0, \tag{3.75}$$

i. e., the mean value on γ of the circumferential bending moment M_τ is zero in a plate under torsion, having a hole of arbitrary shape.

If the plate is subjected to bending by moments $M_x^{(\infty)}$, only ($M_y^{(\infty)} = 0$), it follows from (3.74) that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{M_\tau}{M_x^{(\infty)}} d\theta = 1, \tag{3.76}$$

i. e., in this case the mean value on γ of the coefficient of stress concentration is unity in a plate having a hole of arbitrary shape.

Equations (3.71) to (3.76), derived above, can be used to test the correctness of solutions of the corresponding problems, and to estimate the accuracy of approximate solutions obtained.

§ 4. CIRCULAR ANNULAR PLATE WITH REINFORCED INNER AND FREE OUTER EDGE. SOLUTION BY FOURIER SERIES*

Consider a circular annular plate, reinforced along the inner edge Γ of radius R by a thin elastic rib of uniform cross section. Let L be the outer concentric edge of the plate, of radius R_1. The plate carries an arbitrary transverse load $q(r, \theta)$, including concentrated forces and couples, and an arbitrary load along L and Γ (Figure 26).

The functions $\varphi(z)$ and $\psi(z)$, which define the additional deflections of the plate, due to the stiffening rib, are holomorphic and single-valued in the domain of the plate, and can therefore be represented by Laurent series:

$$\varphi(z) = \sum_{-\infty}^{+\infty} \beta_n \frac{z^n}{R^n}, \qquad \psi(z) = \sum_{-\infty}^{+\infty} \alpha_n \frac{z^n}{R^n}, \tag{3.77}$$

* The same solution was obtained in a slightly different way in /145/.

where α_n and β_n are generally complex constants. Since the deflections of the plate are single-valued, α_{-1} must be real.

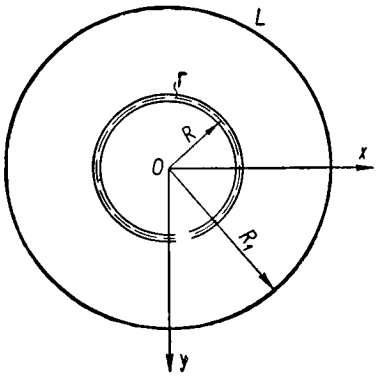


FIGURE 26.

The boundary conditions are, according to (3.1), (3.2), and (1.65)

$$\left.\begin{aligned} -\varkappa\overline{\varphi(t)}+t\bar{\varphi}'(t)+\psi(t)&=(\varkappa+1)\overline{g(t)} \\ \overline{\varphi(t)}+\bar{t}\varphi'(t)+\psi(t)&=P(t)-R(t)\end{aligned}\right\}\text{on }\Gamma, \tag{3.78}$$

$$-\varkappa\overline{\varphi(t_1)}+\bar{t}_1\varphi'(t_1)+\psi(t_1)=-iC_1\bar{t}_1+C_2 \text{ on } L, \tag{3.80}$$

where $t_1=R_1e^{i\theta}=R_1\sigma$ is the complex coordinate (affix) of the point on L, and $t=R\sigma$ is the affix of the point on Γ; C_1 is a real, C_2 a complex constant.

Expanding the function $R(t)$ in a complex Fourier series, we obtain

$$R(t)=\left(\frac{\partial w^0}{\partial t}\right)_\Gamma-\frac{C_2}{2i}=-2\sum_{-\infty}^{+\infty}b_n\sigma^n-\frac{C_2}{2i}. \tag{3.81}$$

The complex coefficients b_n are known; since the basic deflections w^0 are real and periodic, the coefficient b_{-1} is real by virtue of (2.261).

The unknown functions $P(t)$ and $g(t)$, entering the right-hand sides of (3.78) and (3.79), are related by (3.6).

Substituting (3.77) in (3.78), we express $g(t)$ through the unknown coefficients α_k, β_k:

$$g(t)=\sum_{-\infty}^{+\infty}a_n\sigma^n, \tag{3.82}$$

where

$$a_n=-\frac{1-\nu}{4}\left[\frac{3+\nu}{1-\nu}\beta_n+(n-2)\bar{\beta}_{-(n-2)}-\bar{\alpha}_{-n}\right]$$
$$(n=0,\pm1,\pm2,\ldots). \tag{3.83}$$

Substitution of (3.82) in (3.6) yields

$$P(t) = K \ln \sigma + 2 \sum_{-\infty}^{+\infty} B_n \sigma^n, \tag{3.84}$$

where, in accordance with (2.260), [cf. (2.245)]

$$B_n = \left(\frac{1}{\delta_2} + \frac{1}{\delta_1}\right) \frac{\bar{a}_{-n}}{n} - \left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right) \frac{a_{n+2}}{n} \quad (n = \pm 1, \pm 2, \ldots), \tag{3.85}$$

$$\left.\begin{aligned} B_0 &= \left(\frac{1}{\delta_2} + \frac{1}{\delta_1}\right) \sum_{-\infty}^{+\infty *} \frac{\bar{a}_n}{n} + \left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right) \sum_{-\infty}^{+\infty} \frac{a_n}{n-2} \\ K &= 2\left[\bar{a}_0 \left(\frac{1}{\delta_2} + \frac{1}{\delta_1}\right) - a_2 \left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right)\right] \end{aligned}\right\}. \tag{3.86}$$

The sums $\sum_{-\infty}^{+\infty *}$ and $\sum_{-\infty}^{+\infty '}$ include no terms corresponding to $n = 0$ and $n = 2$, respectively.

Since the function $P(t)$ defined by (3.84) is single-valued, we must have $K = 0$, i.e.,

$$\bar{a}_0 \left(\frac{1}{\delta_2} + \frac{1}{\delta_1}\right) - a_2 \left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right) = 0$$

or, using (3.83),

$$\left(\frac{1}{\delta_2} + \frac{1}{\delta_1}\right)\left(\frac{3+\nu}{1-\nu}\bar{\beta}_0 - 2\beta_2 - \alpha_0\right) - \left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right)\left(\frac{3+\nu}{1-\nu}\beta_2 - \bar{\alpha}_{-2}\right) = 0. \tag{3.87}$$

Substituting (3.77), (3.81), and (3.84) in (3.79) and (3.80), and equating the coefficients of equal powers of σ, we obtain

$$\beta_2 + \bar{\alpha}_{-2} - 2\bar{B}_{-2} = 2\bar{b}_{-2}, \tag{3.88}$$

$$-\varkappa\beta_2\eta^2 + \bar{\alpha}_{-2}\eta^{-2} = 0, \tag{3.89}$$

$$\beta_1 - \varkappa\bar{\beta}_1 + \alpha_{-1}\eta^{-2} = -iC_1R_1\eta^{-1}, \tag{3.90}$$

$$\bar{\beta}_1 + \beta_1 + \alpha_{-1} - 2B_{-1} = 2b_{-1}, \tag{3.91}$$

$$\left.\begin{aligned} &\bar{\beta}_{-n} + (n+2)\beta_{n+2} + \alpha_n - 2B_n = 2b_n \\ &-\varkappa\bar{\beta}_{-n}\eta^{-n} + (n+2)\beta_{n+2}\eta^{n+2} + \alpha_n\eta^n = 0 \end{aligned}\right\} (n = 1, 2, \pm 3, \pm 4, \ldots), \tag{3.92, 3.93}$$

$$-\varkappa\bar{\beta}_0 + 2\beta_2\eta^2 = C_2, \tag{3.94}$$

$$\bar{\beta}_0 + 2\beta_2 - 2B_0 = 2b_0 + \frac{C_3}{2i}, \tag{3.95}$$

where $\eta = \frac{R_1}{R}$. Due to the indeterminacy of $\psi(z)$, we may assume that $\alpha_0 = 0$.

Solving the system (3.87), (3.88), (3.89), we obtain

$$\left.\begin{aligned}\beta_2 &= \frac{2(\delta_1+\delta_2)\bar{b}_{-2}}{(1+\varkappa\eta^4)(\delta_1+\delta_2)+(1-\nu)\varkappa\eta^4-(3+\nu)}\\ \beta_0 &= \left[\frac{2(1-\nu)}{3+\nu}+\frac{\delta_1-\delta_2}{\delta_1+\delta_2}(1-\eta^4)\right]\bar{\beta}_2, \quad \bar{\alpha}_{-2}=\varkappa\eta^4\beta_2\end{aligned}\right\}. \tag{3.96}$$

Since b_{-1} and α_{-1} are real, we find from (3.91) that B_{-1} is also real. From (3.90) we first find that $C_1 = 0$ [sic]. Solving (3.90) and (3.91) simultaneously, we obtain

$$\begin{aligned}&\beta_1 = -\frac{2\delta_1}{\Delta_2}b_{-1}, \qquad \alpha_{-1}=-\frac{2\delta_1}{\Delta_2}(\varkappa-1)\eta^2 b_{-1},\\ &\Delta_2=(1-\varkappa)\eta^2(1-\nu+\delta_1)+2(1+\nu-\delta_1).\end{aligned} \tag{3.97}$$

If we write $n-1$ and $-(n+1)$ for n in (3.92) and (3.93), we obtain the following four equations, which serve for determining the coefficients $\bar{\beta}_{-(n-1)}$, β_{n+1}, α_{n-1}, $\bar{\alpha}_{-(n+1)}$ $(n \geqslant 2)$:

$$\left.\begin{aligned}&\beta_{n+1}-(n-1)\bar{\beta}_{-(n-1)}+\bar{\alpha}_{-(n+1)}-2\bar{B}_{-(n+1)}=2\bar{b}_{-(n+1)}\\ &-\varkappa\beta_{n+1}\eta^{n+1}-(n-1)\eta^{-(n-1)}\bar{\beta}_{-(n-1)}+\eta^{-(n+1)}\bar{\alpha}_{-(n+1)}=0\\ &(n+1)\beta_{n+1}+\bar{\beta}_{-(n-1)}+\alpha_{n-1}-2B_{n-1}=2b_{n-1}\\ &(n+1)\beta_{n+1}\eta^{n+1}-\varkappa\bar{\beta}_{-(n-1)}\eta^{-(n-1)}+\eta^{n-1}\alpha_{n-1}=0\end{aligned}\right\}. \tag{3.98}$$

It remains to prove that system (3.98) has always a solution for $n \geqslant 2$. We substitute for $\bar{\alpha}_{-(n+1)}$ and α_{n-1} from the second and fourth equations in the remaining equations, and obtain a system of two equations in two unknowns β_{n+1} and $\bar{\beta}_{-(n-1)}$. Its determinant is

$$\Delta_n = \frac{1}{2(n^2-1)}\left[A^{(n)}+B^{(n)}\frac{1}{\delta_1}+C^{(n)}\frac{1}{\delta_2}+D^{(n)}\frac{1}{\delta_1\delta_2}\right]>0,$$

where

$$A^{(n)} = 2(n^2-1)\left\{(n^2-1)(\eta^2-1)^2+\left[1+\frac{3+\nu}{1-\nu}\eta^{2(n+1)}\right]\times\right.$$
$$\left.\times\left[1+\frac{3+\nu}{1-\nu}\eta^{-2(n-1)}\right]\right\}>0,$$

$$B^{(n)}=\frac{2(3+\nu)}{1-\nu}[(2n+1+\nu)\eta^{2(n+1)}-(2n-1-\nu)\eta^{-2(n-1)}-$$
$$-(3+\nu)\eta^4]+2n^2(3+\nu)-2\eta^2(n^2-1)[\eta^2(1-\nu)+2(1+\nu)]>0,$$

$$C^{(n)}=\frac{2(3+\nu)}{1-\nu}[(2n+1+\nu)\eta^{2(n+1)}-(2n-1-\nu)\eta^{-2(n-1)}-$$
$$-(3+\nu)\eta^4]-2\eta^4(n^2-1)(1-\nu)+4(n^2-1)(3-\nu)\eta^2-2(5-\nu)n^2+16>0,$$

$$D^{(n)}=2(3+\nu)^2(\eta^{2(n+1)}-1)(1-\eta^{-2(n-1)})-2(n^2-1)(1-\nu)^2(\eta^2-1)^2>0.$$

It follows that system (3.98) defines uniquely the coefficients α_n and $\beta_n (n = 0, \pm 1, \pm 2, \ldots)$ of $\varphi(z)$ and $\psi(z)$ in (3.77), so that the problem formulated is completely solved. Equations (3.94) and (3.95), which have not been used so far, serve for determining the complex constants C_2 and C_3, respectively; we shall not do this since they are not necessary to us.

If we replace R_1 by R and η by η^{-1} in the above solution, we obtain the additional deflections and stresses in an annular plate whose outer edge is reinforced by an elastic rib. In the limit $\delta_1 = \delta_2 = \infty$ we obtain the solution for a rigidly clamped inner contour of the plate.

The problem of a plate with both outer and inner edge reinforced by thin elastic ribs presents no difficulties. We must then replace (3.80) by (3.1) and (3.2).

Axisymmetrical bending

In the case of axisymmetrical bending we have

$$\left(\frac{\partial w^0}{\partial t}\right)_\Gamma = \frac{1}{2}\left(\frac{dw^0}{dr}\right)_\Gamma \sigma^{-1}$$

so that all the coefficients b_n in (3.81) vanish except

$$b_{-1} = -\frac{1}{4}\left(\frac{dw^0}{dr}\right)_\Gamma.$$

We obtain therefore from (3.97):

$$\beta_1 = \frac{\delta_1}{2\Delta_3}\left(\frac{dw^0}{dr}\right)_\Gamma, \qquad \alpha_{-1} = -\frac{\delta_1}{2\Delta_2}(1-\varkappa)\eta^2\left(\frac{dw^0}{dr}\right)_\Gamma$$

while (3.77) becomes

$$\varphi(z) = \frac{\delta_1}{2\Delta_2}\left(\frac{dw^0}{dr}\right)_\Gamma \cdot \frac{z}{R}, \quad \psi(z) = \frac{(\varkappa-1)\eta^2\delta_1}{2\Delta_2}\left(\frac{dw^0}{dr}\right)_\Gamma \cdot \frac{R}{z}.$$

From [cf. (1.22)]

$$w^1 = 2\,\mathrm{Re}[\bar{z}\varphi(z) + \chi(z)],$$

we obtain the additional deflections:

$$w^1 = \mu_1(r)\lambda_3(\delta_1, k) R_1 \left(\frac{dw^0}{dr}\right)_\Gamma + C^*, \tag{3.99}$$

where

$$\left.\begin{aligned} k &= \eta^{-1} = \frac{R}{R_1} \\ \mu_1(r) &= 4\left[\left(1-\frac{r^2}{R_1^2}\right)\frac{1}{\varkappa-1} - \ln\frac{r}{R_1}\right] \\ \lambda_3(\delta_1, k) &= \frac{\delta_1(1-\varkappa)}{(1-\varkappa)(\delta_1+1-\nu)+2k^2(1+\nu-\delta_1)}\cdot\frac{k}{4} \end{aligned}\right\}. \tag{3.100}$$

Since $\varkappa = \frac{3+\nu}{1-\nu}$, we find that the function $\mu_1(r)$ in (3.100) is identical with $\mu_1(r, k)$ in (2.273) for $r \geqslant R$.

The constant C^* is determined from the condition that the deflections vanish on the support contour. If the plate is supported at the contour $r = R_1$, we have $C^* = 0$.

The additional bending moments in the plate are [cf. (2.275), (2.276)]

$$\left.\begin{aligned} M_r &= -\frac{(1-\nu^2)D\delta_1}{R\Delta_*}\left(\frac{R_1^2}{r^2}-1\right)\left(\frac{dw^0}{dr}\right)_\Gamma \\ M_\theta &= \frac{(1-\nu^2)D\delta_1}{R\Delta_*}\left(\frac{R_1^2}{r^2}+1\right)\left(\frac{dw^0}{dr}\right)_\Gamma \end{aligned}\right\}, \tag{3.101}$$

where

$$\Delta_* = [(1-\nu^2)(\eta^2-1)+\delta_1(1+\nu)\eta^2+(1-\nu)\delta_1] > 0, \quad \eta = \frac{R_1}{R}.$$

Adding the additional deflections w^1 to the known basic deflection of the nonreinforced axisymmetrically loaded annular plate, we obtain the total deflections of the annular plate with reinforced inner edge.

We present solutions of some simple examples.

Bending of an annular plate, due to moments acting at the edges. An annular plate is bent by applying moments M = const. along the outer contour $r=R_1$, and moments m = const. along the inner contour $r=R$, which is stiffened by a thin rib (Figure 27).

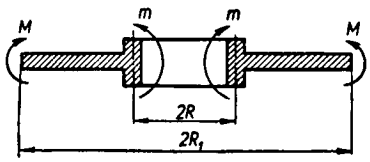


FIGURE 27.

From the solution for a nonreinforced plate /169/ we obtain the following expression for the total deflection:

$$w = w^1 + w^0 = \frac{R^2}{2D\Delta_*}\left\{[m(1-\nu) - M\eta^2(1-\nu+\delta_1)]\frac{r^2}{R^2} + \right.$$

$$\left. + 2\eta^2[m(1+\nu) - M(1+\nu-\delta_1)]\ln\frac{r}{R}\right\} + \text{const}.$$

The corresponding bending moments are

$$M_r = -\frac{1}{\Delta_*}\left\{m(1-\nu^2)\left(1-\frac{R_1^2}{r^2}\right) + \right.$$

$$\left. + M\eta^2\left[(1-\nu)(1+\nu-\delta_1)\frac{R^2}{r^2} - (1+\nu)(1-\nu+\delta_1)\right]\right\},$$

$$M_\theta = -\frac{1}{\Delta_*}\left\{m(1-\nu^2)\left(1+\frac{R_1^2}{r^2}\right) - \right.$$

$$\left. - M\eta^2\left[(1-\nu)(1+\nu-\delta_1)\frac{R^2}{r^2} + (1+\nu)(1-\nu+\delta_1)\right]\right\}.$$

If no external forces and moments act on the stiffening rib, ($m = 0$), the optimum value of δ_1, for which

$$M_r = M_\theta = M$$

over the entire plate, is

$$\delta_1^{\text{opt}} = (1+\nu).$$

The stress concentration near the hole is then completely eliminated, and the plate behaves like a solid disk under the same load.

Reinforcement of the inner plate edge by a rib of optimum rigidity $\delta_1^{\text{opt}} = (1+\nu)$ causes the maximum normal stresses to decrease by more than 50%.

The ratio of the maximum stresses

$$\eta_* = \frac{(\sigma_{\max})_{\delta_1=0}}{(\sigma_{\max})_{\delta_1=1+\nu}}$$

is given below as a function of $\eta = \frac{R_1}{R}$.

η	1.25	1.5	2.0	3.0	4.0	5.0	20.0	∞
η_*	5.556	3.600	2.667	2.250	2.133	2.083	2.005	2.000

Bending of an annular plate by application of shearing forces. The annular plate is supported along the inner reinforced edge $r=R$, and subjected to a uniformly distributed load p = const. along the outer edge $r=R_1$ (Figure 28).

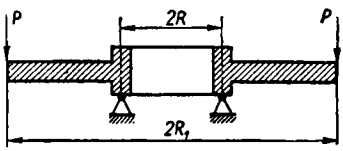


FIGURE 28.

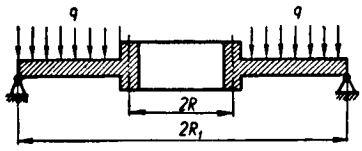


FIGURE 29.

We obtain the total deflections from the known solution for the nonreinforced plate /136/:

$$w = \frac{pR_1^3}{8\eta^2 D\Delta_*}\left\{4\eta^2[(1+\nu-\delta_1)(1+\nu)\ln\eta-\delta_1]\ln\frac{R}{r}+\right.$$

$$+2[\eta^2(1+\nu)(\nu-1-\delta_1)+(1-\nu)(1+\nu-\delta_1)]\frac{r^2}{R^2}\ln\frac{R}{r}+$$

$$+\left(1-\frac{r^2}{R^2}\right)[\eta^2(3+\nu)(1-\nu+\delta_1)-(1-\nu)(3+\nu-\delta_1)+$$

$$\left.+2\eta^2(1+\nu)(1-\nu+\delta_1)\ln\eta]\right\}. \qquad (3.102)$$

The bending moments and shearing forces in a plate with stiffening rib are

$$M_r = \frac{-pR_1}{2\Delta_*}\left\{\delta_1(1-\nu)\left(1-\eta^2\frac{R^2}{r^2}\right)+(1+\nu)[\eta^2(1+\nu)(1-\nu+\delta_1)-\right.$$

$$-(1-\nu)(1+\nu-\delta_1)]\ln\frac{r}{R}-\eta^2(1+\nu)\left[(1+\nu)(1-\nu+\delta_1)-\right.$$

$$\left.\left.-(1-\nu)(1+\nu-\delta_1)\frac{R^2}{r^2}\right]\ln\eta\right\},$$

$$M_\theta = \frac{-pR_1}{2\Delta_*}\left\{\nu\delta_1(1-\nu)(1-\eta^2)+(1+\nu)[\eta^2(1+\nu)(1-\nu+\delta_1)-\right.$$

$$-(1-\nu)(1+\nu-\delta_1)]\ln\frac{r}{R}-\eta^2(1+\nu)\left[(1+\nu)(1-\nu+\delta_1)+\right.$$

$$+(1-\nu)(1+\nu-\delta_1)\frac{R^2}{r^2}\left.\right]\ln\eta-(1+\nu)(1-\nu)^2(\eta^2-1)-$$

$$\left.-\eta^2\delta_1(1-\nu)\left(1-\frac{R^2}{r^2}\right)\right\}, \quad N_r = pR_1r^{-1}, \quad N_\theta = H_{r\theta} = 0.$$

Table 6 gives values of $\sigma_{max}\Big/\frac{2\pi pR_1}{h^2}$ for some values of $\eta = \frac{R_1}{R}$ for $\nu = 0.3$ and $0 < \delta_1 < \infty$.

Table 6 shows that by reinforcing the inner edge of the plate, we can, at given load, reduce considerably the calculated stresses in the plate. A detailed analysis of the solution is given in /145/. For $\delta_1 = 1+\nu$ the plate behaves like a solid disk under the same load.

TABLE 6

δ_1	η					
	1.25	1.5	2	3	4	5
0	1.100	1.24	1.48	1.87	2.17	2.41
1	0.231	0.395	0.623	0.919	1.12	1.27
1.3	0.194	0.345	0.556	0.830	1.02	1.16
2	0.208	0.370	0.612	0.932	1.15	1.31
5	0.219	0.402	0.689	1.08	1.34	1.54
10	0.223	0.415	0.720	1.14	1.42	1.64
∞	0.227	0.428	0.753	1.20	1.51	1.75

The method of dimensioning a rib of the required rigidity will be explained in § 8 of this chapter.

Bending of a uniformly loaded simply supported annular plate. An annular plate is simply supported along the outer edge $r=R_1$ and subjected to a uniformly distributed load q = const. The inner plate edge $r=R$ is reinforced by an elastic rib on which no external load acts (Figure 29).

We obtain the total deflections from the known solution for the non-reinforced plate /136/:

$$w = w^1 + w^0 = \frac{qr^4}{64D} + D_1 \ln \frac{r}{R} + D_2 \frac{r^2}{R^2} \ln \frac{r}{R} + D_3 + D_4 \frac{r^2}{R^2}, \qquad (3.103)$$

where

$$\left.\begin{aligned}
D_1 &= \frac{qR^4\eta^2}{16D\Delta_*}\{(1+\nu-\delta_1)[(3+\nu)(2-\eta^2)+4(1+\nu)\ln\eta]- \\
&\quad -(1+\nu)(3+\nu-\delta_1)\} \\
D_2 &= -\frac{qR^4}{8D} \\
D_3 &= -\frac{qR^4\eta^2}{64D\Delta_*}\{[16+8(1+\nu)\ln\eta-(5+\nu)\eta^2][\eta^2(\delta_1+1-\nu)+ \\
&\quad +2(1+\nu-\delta_1)\ln\eta]+[(\nu-1)-(1+\nu)\ln\eta][2(3+\nu-\delta_1)+ \\
&\quad +\eta^2(5-3\nu+3\delta_1)]\} \\
D_4 &= \frac{qR^4}{32D\Delta_*}\{\eta^2(\delta_1+1-\nu)[(3+\nu)(2-\eta^2)+4(1+\nu)\ln\eta+ \\
&\quad +(1-\nu)(\delta_1-3-\nu)\}
\end{aligned}\right\} \cdot \qquad (3.104)$$

TABLE 7

δ_1	η					
	1.25	1.5	2	3	4	5
0	0.592	0.976	1.443	1.881	2.082	2.192
1	0.124	0.310	0.607	0.925	1.075	1.157
1.3	0.107	0.271	0.541	0.836	0.976	1.052
2	0.111	0.291	0.596	0.938	1.103	1.193
5	0.118	0.317	0.671	1.086	1.289	1.401
10	0.120	0.326	0.701	1.146	1.366	1.487
20	0.121	0.332	0.717	1.179	1.408	1.534
∞	0.122	0.336	0.734	1.214	1.453	1.585

Setting in (3.104) $\delta_1 = \infty$ we obtain the solution for the case of the inner plate edge being rigidly clamped /136/.

The bending moments in the plate are

$$\left.\begin{aligned} M_r &= \frac{qR^2}{16}\left[-(3+\nu)\frac{r^2}{R^2} + 4(1+\nu)\ln\frac{r}{R} + 2(3+\nu) - \right. \\ &\qquad \left. - 4(1+\nu)C_4^* + 2(1-\nu)C_1^*\frac{R^2}{r^2}\right] \\ M_\theta &= \frac{qR^2}{16}\left[-(1+3\nu)\frac{r^2}{R^2} + 4(1+\nu)\ln\frac{r}{R} + 2(1+3\nu) - \right. \\ &\qquad \left. - 4(1+\nu)C_4^* - 2(1-\nu)C_1^*\frac{R^2}{r^2}\right] \\ C_4^* &= \frac{8D}{qR^4}D_4, \quad C_1^* = \frac{8D}{qR^4}D_1 \end{aligned}\right\} \qquad (3.105)$$

An analysis of (3.105) shows that for $\delta_1 < (1+\nu)$ the moment M_θ is greater on $r = R$, while for $\delta_1 > (1+\nu)$ the moment M_r is greater. For $\delta_1 = 1+\nu$ we have $M_r = M_\theta$ on $r = R$.

Table 7 gives values of $\sigma_{max}\left|\frac{qR_1^2}{h^2}\right.$ for some values of η for $\nu = 0.3$ and $0 < \delta_1 < \infty$.

When $\delta_1 = 1+\nu$ the stresses in the plate are minimum. The calculated stresses in the plate are reduced by more than 50% when the inner plate edge is reinforced by an elastic rib.

§ 5. CIRCULAR ANNULAR PLATE WITH REINFORCED INNER AND RIGIDLY CLAMPED OUTER EDGE. SOLUTION BY FOURIER SERIES

A circular annular plate is reinforced along the inner contour Γ ($r = R$) and rigidly clamped along the outer contour L ($r = R_1$). The load is arbitrary (Figure 30).

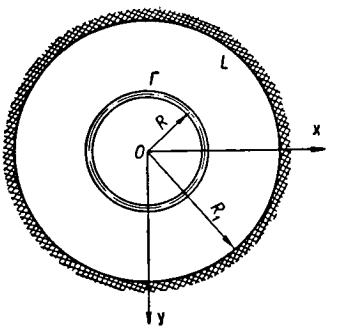


FIGURE 30.

We represent the functions $\varphi(z)$ and $\psi(z)$ in Laurent series, the origin of coordinates being at the center of the hole.

$$\left.\begin{aligned}\varphi(z) &= \sum_{-\infty}^{+\infty} \beta_n \frac{z^n}{R^n},\\ \psi(z) &= \sum_{-\infty}^{+\infty} \alpha_n \frac{z^n}{R^n},\end{aligned}\right\} \tag{3.106}$$

where α_{-1} and β_{-1} are real while all the other coefficients are generally complex.

The boundary conditions in this case coincide with (3.78) to (3.80), if in (3.80) we write $\varkappa = -1$ and $C_1 = C_2 = 0$. Substituting these values in (3.88) to (3.91), (3.94), (3.95), and (3.98), we can write down directly the system of equations for determining the coefficients in (3.106);

$$\left(\frac{1}{\delta_2}+\frac{1}{\delta_1}\right)\left(\frac{3+\nu}{1-\nu}\bar{\beta}_0 - 2\beta_2 - \alpha_0\right) - \left(\frac{1}{\delta_2}-\frac{1}{\delta_1}\right)\left(\frac{3+\nu}{1-\nu}\beta_2 - \bar{\alpha}_{-2}\right) = 0, \tag{3.107}$$

$$\beta_2 + \bar{\alpha}_{-2} - 2\bar{B}_{-2} = 2\bar{b}_{-2}, \tag{3.108}$$

$$\beta_2\eta^2 + \bar{\alpha}_{-2}\eta^{-2} = 0, \tag{3.109}$$

$$\bar{\beta}_0 + 2\beta_2\eta^2 + \alpha_0 = 0, \tag{3.110}$$

$$2\beta_1 + \alpha_{-1}\eta^{-2} = 0, \tag{3.111}$$

$$2\beta_1 + \alpha_{-1} - 2B_{-1} = 2b_{-1}, \tag{3.112}$$

$$\beta_{n+1} - (n-1)\bar{\beta}_{-(n-1)} + \bar{\alpha}_{-(n+1)} - 2\bar{B}_{-(n+1)} = 2\bar{b}_{-(n+1)}, \tag{3.113}$$

$$\eta^{n+1}\beta_{n+1} - (n-1)\eta^{-(n-1)}\bar{\beta}_{-(n-1)} + \eta^{-(n+1)}\bar{\alpha}_{-(n+1)} = 0, \tag{3.114}$$

$$(n+1)\beta_{n+1} + \bar{\beta}_{-(n-1)} + \alpha_{n-1} - 2B_{n-1} = 2b_{n-1}, \tag{3.115}$$

$$(n+1)\beta_{n+1}\eta^{n+1} + \bar{\beta}_{-(n-1)}\eta^{-(n-1)} + \eta^{n-1}\alpha_{n-1} = 0, \tag{3.116}$$

$$\bar{\beta}_0 + 2\beta_2 - 2B_0 = 2b_0 + \frac{C_3}{2i}. \tag{3.117}$$

Solving (3.107) to (3.110) simultaneously, we find:

$$\left.\begin{aligned}\bar{\alpha}_{-2} &= \frac{2\eta^4\bar{b}_{-2}(\delta_1+\delta_2)}{(\eta^4-1)(\delta_1+\delta_2)+(3+\nu)+\eta^4(1-\nu)}\\ \beta_2 &= -\frac{2\bar{b}_{-2}(\delta_1+\delta_2)}{(\eta^4-1)(\delta_1+\delta_2)+(3+\nu)+\eta^4(1-\nu)}\\ \alpha_0 &= -2\eta^2\beta_2 - \bar{\beta}_0,\\ \bar{\beta}_0 &= \frac{(1-\nu)\beta_2}{4(\delta_1+\delta_2)}\left[2(1-\eta^2)(\delta_1+\delta_2)+(\delta_1-\delta_2)\left(\frac{3+\nu}{1-\nu}+\eta^4\right)\right]\end{aligned}\right\}. \tag{3.118}$$

We obtain from (3.111) and (3.112)

$$\beta_1 = -\frac{2b_{-1}\delta_1}{\Delta_3}, \qquad \alpha_{-1} = \frac{4\eta^2 b_{-1}\delta_1}{\Delta_3}, \tag{3.119}$$

where

$$\Delta_3 = 2\eta^2(1-\nu+\delta_1)+2(1+\nu-\delta_1). \tag{3.120}$$

Equations (3.113) to (3.116) have a single-valued solution for $n \geqslant 2$ and serve to determine the coefficients β_{n+1}, $\bar{\beta}_{-(n-1)}$, α_{n-1}, $\bar{\alpha}_{-(n+1)}$ $(n \geqslant 2)$.

Substituting the known coefficients entering in (3.106), we then determine C_3 from (3.117).

Replacing η by η^{-1} in the above solution, we obtain the additional deflections and stresses of an annular plate whose inner edge is rigidly clamped, the outer edge being reinforced by an elastic rib.

In particular, when $\delta_1 = \delta_2 = \infty$, we obtain the solution for a plate with perfectly rigid central hub.

Bending of a plate by application of a moment

Let the inner edge of the plate be subjected to shearing forces whose distribution is given by $p = \frac{M}{\pi R^2}\cos\theta$. The principal vector (force resultant) of this load is zero, and the principal moment equals M (Figure 31).

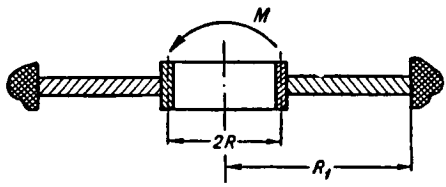


FIGURE 31.

The basic deflection is in this case

$$w^0 = \frac{MR_1}{8\pi D}\left[(1-2c)\frac{r}{R_1}+(c-1)\frac{r^3}{R_1^3}+2\frac{r}{R_1}\ln\frac{r}{R}+c\frac{R_1}{r}\right]\cos\theta, \tag{3.121}$$

where

$$c = \frac{3+\nu-\eta^2(1+\nu)}{3+\nu+\eta^4(1-\nu)}. \tag{3.122}$$

Expanding the expression $\left(\frac{\partial w^0}{\partial t}\right)_\Gamma$ in (3.81) in a Fourier series, we obtain

$$b_0 = \frac{M}{16\pi D\eta^2}[\eta^2\ln\eta+(1-c)(1-\eta^2)],\ b_{-2} = -\frac{M(\eta^2-1)^2}{16\pi D[(3+\nu)+\eta^4(1-\nu)]}. \tag{3.123}$$

All the remaining coefficients b_n vanish.

Substituting (3.123) in (3.118), we obtain the coefficients α_0, β_0, α_{-2}, β_2 of the functions $\varphi(z)$ and $\psi(z)$ in (3.106). We then determine from (2.142) the

additional deflections w^1 due to the rib:

$$w^1 = B\left(2\frac{r}{R_1} - \frac{r^3}{R_1^3} - \frac{R_1}{r}\right)\cos\theta,$$

where

$$B = -2\eta^2 R_1\beta_2 = -\frac{MR_1}{4\pi D}\cdot\frac{\eta^2(\eta^2-1)^2(\delta_1+\delta_2)}{[(\eta^4-1)(\delta_1+\delta_2)+(3+\nu)+}$$
$$+\eta^4(1-\nu)][(3+\nu)+\eta^4(1-\nu)].$$

In particular, letting $(\delta_1+\delta_2)\to\infty$, we obtain the known solution /43, 213/ for a plate whose center forms a perfectly rigid core to which the bending moment M is applied.

The angle through which the plane of the stiffening rib rotates is

$$\varphi = -\left(\frac{w}{r\cos\theta}\right)_{r=R} = -\left(\frac{w^0+w^1}{r\cos\theta}\right)_{r=R} = C_*\frac{M}{Eh^3}.$$

Table 8 gives C_* for some values of η and $(\delta_1+\delta_2)$, with $\nu = 0.3$. It is seen that even a very small rib (e. g., for which $\delta_1+\delta_2 = 1$) increases considerably the rigidity of the plate. The elasticity of the plate-edge clamping should be taken into account.

TABLE 8

η^{-1}	$\delta_1+\delta_2$					
	0	1	2	5	10	∞
0.5	0.4046	0.2400	0.1864	0.1334	0.1095	0.0809
0.6	0.1835	0.1188	0.0934	0.0655	0.0520	0.0350
0.7	0.0643	0.0468	0.0382	0.0271	0.0210	0.0125
0.8	0.0152	0.0125	0.0108	0.0081	0.0063	0.0032

Axisymmetrical bending

Setting $\varkappa = -1$ in (3.99) and (3.100), we obtain the additional deflections w^1, due to a stiffening rib, for an arbitrary axisymmetrical load (the constant C^* is determined from the condition that $w^1 = 0$ for $r = R_1$):

$$w^1 = \mu_2(r)\lambda_4(\delta_1, k)R_1\left(\frac{dw^0}{dr}\right)_\Gamma, \qquad (3.124)$$

where

$$\mu_2(r) = \frac{1}{2}\left(\frac{r^2}{R_1^2}-1\right) - \ln\frac{r}{R_1},$$

$$\lambda_4(\delta_1, k) = \frac{k\delta_1}{(1-\nu+\delta_1)+k^2(1+\nu-\delta_1)}, \quad k = \eta^{-1} = \frac{R}{R_1}.$$

The additional bending moments in the plate are

$$\left.\begin{aligned} M_r &= -\frac{D\delta_1}{R\Delta_{**}}\left[(1-\nu)\frac{R_1^2}{r^2}+(1+\nu)\right]\left(\frac{dw^0}{dr}\right)_\Gamma \\ M_\theta &= \frac{D\delta_1}{R\Delta_{**}}\left[(1-\nu)\frac{R_1^2}{r^2}-(1+\nu)\right]\left(\frac{dw^0}{dr}\right)_\Gamma \end{aligned}\right\}, \tag{3.125}$$

where

$$\Delta_{**} = \eta^2(\delta_1 + 1 - \nu) + (1 + \nu - \delta_1). \tag{3.126}$$

We shall now give some simple examples.

Bending of an annular plate under a uniformly distributed load. A circular annular plate is rigidly clamped along the outer edge and subjected to a uniformly distributed load q = const. The inner edge of the plate is reinforced by an elastic rib on which no external load acts (Figure 32).

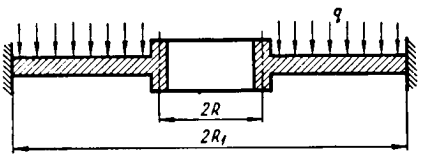


FIGURE 32.

From the known solution for the nonreinforced plate /75/, we obtain the total deflection

$$w = w^1 + w^0 = \frac{qr^4}{64D} + D_1 \ln\frac{r}{R} + D_2\frac{r^2}{R^2}\ln\frac{r}{R} + D_3 + D_4\frac{r^2}{R^2}, \tag{3.127}$$

where

$$\left.\begin{aligned} D_1 &= -\frac{\eta^2 qR^4}{16D\Delta_{**}}\left[(1+\nu-\delta_1)(\eta^2 - 4\ln\eta) + (\delta_1 + 1 - \nu)\right] \\ D_2 &= -\frac{qR^4}{8D} \\ D_3 &= \frac{\eta^2 qR^4}{64D\Delta_{**}}\left[\eta^4(1-\nu+\delta_1) + \eta^2(3\nu - 5 - 3\delta_1) + 4\eta^2(1+\nu-\delta_1)\ln\eta + \right. \\ &\quad \left. + 4(3+\nu-\delta_1)\ln\eta - 2(3+\nu-\delta_1) - 16(1+\nu-\delta_1)\ln^2\eta\right] \\ D_4 &= \frac{qR^4}{32D\Delta_{**}}\left[(3+\nu-\delta_1) + (1-\nu+\delta_1)(2\eta^2 - \eta^4 + 4\eta^2\ln\eta)\right] \end{aligned}\right\}. \tag{3.128}$$

The bending moments in the plate are given by (3.105), where D_1 and D_4 are defined by (3.128).

Table 9 gives the magnitude $\sigma_{max}\left|\frac{qR_1^2}{h^2}\right.$ for some values of η and δ_1 with ν=0.3. It is seen that by reinforcing the inner edge of the plate we can

reduce the calculated stresses. The smaller η, the more effective is the stiffening rib. The minimum values of σ_{max} correspond to an absolutely rigid core ($\delta_1 = \infty$).

TABLE 9

δ_1	η					
	1.25	1.5	2	3	4	5
0	0.105	0.259	0.480	0.657	0.710	0.730
1	0.099	0.236	0.430	0.600	0.665	0.695
2	0.095	0.223	0.409	0.581	0.651	0.685
5	0.087	0.207	0.386	0.563	0.639	0.676
10	0.082	0.197	0.375	0.555	0.633	0.672
∞	0.072	0.182	0.361	0.546	0.629	0.668

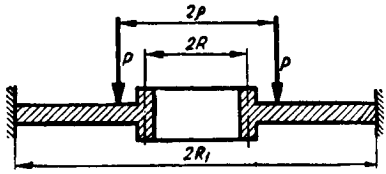


FIGURE 33.

Action of a load distributed along a circle. The total deflections of the annular plate, shown schematically in Figure 33, are /31/:

for $\varrho < r < R_1$

$$w = \frac{pR_1^2\varrho}{8D\Delta_{**}}\left\{-2\Delta_{**}\frac{r^2}{R_1^2}\ln\frac{R_1}{r} + 2\left[(1+\nu-\delta_1)\left(2\ln\frac{R_1}{\varrho}-1\right)+\right.\right.$$
$$+\frac{\varrho^2}{R^2}(\nu-1-\delta_1)\Big]\ln\frac{R_1}{r}+\Big[2(1+\nu-\delta_1)\left(1-\ln\frac{R_1}{\varrho}\right)+$$
$$\left.+(1-\nu+\delta_1)\left(\eta^2+\frac{\varrho^2}{R^2}\right)\right]\left(1-\frac{r^2}{R_1^2}\right)\Big\}; \qquad (3.129)$$

for $R < r < \varrho$

$$w = \frac{pR_1^2\varrho}{8D\Delta_{**}}\left\{\left[4\left(1+\ln\frac{r}{R_1}\right)(1+\nu-\delta_1)+2\eta^2(1-\nu+\delta_1)\frac{r^2+\varrho^2}{R_1^2}\right]\ln\frac{R_1}{\varrho}+\right.$$
$$\left.+\left(1-\frac{\varrho^2}{R_1^2}\right)\left[2(1+\nu-\delta_1)\left(\ln\frac{\varrho}{r}-1\right)+\left(\frac{r^2}{R_1^2}+1\right)(\nu-1-\delta_1)\eta^2\right]\right\}. \qquad (3.130)$$

Substituting $\varrho = R$ in (3.129), we obtain the deflections of a plate loaded by shearing forces p = const. along the inner edge:

$$w = \frac{pR_1^3}{4\eta D\Delta_{**}}\left\{\left[2(1+\nu-\delta_1)\ln\eta-2-\Delta_{**}\frac{r^2}{R_1^2}\right]\ln\frac{R_1}{r}+\right.$$
$$\left.+\left[(1+\nu-\delta_1)(1-\ln\eta)+\frac{1}{2}(1-\nu+\delta_1)(\eta^2+1)\right]\left(1-\frac{r^2}{R_1^2}\right)\right\}. \qquad (3.131)$$

§ 6. METHOD OF EQUIVALENT RIGIDITY

1) We assume as known the deflections $w = w(r, \delta_1)$ of an axisymmetrically loaded circular plate ($r \leqslant R_1$) with concentric stiffening rib of radius R (cf. §§ 12 and 14, Chapter II). The relative flexural rigidity of the rib is $\delta_1 = \frac{A}{RD}$. The plate is loaded only in the annular region $R \leqslant r \leqslant R_1$ (Figure 34,a). We shall show that we can easily find the deflections $w_*(r, \delta_1)$ of an annular plate ($R \leqslant r \leqslant R_1$) under the same load when its inner edge $r = R$ is reinforced by a thin elastic rib whose rigidity is also δ_1 (Figure 34,b).

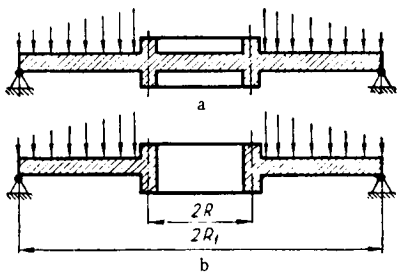


FIGURE 34.

In fact, we conclude from (2.51) that in this case the central disk ($0 \leqslant r \leqslant R$) is equivalent to a stiffening rib whose relative flexural rigidity is $\delta_1 = 1 + \nu$. We therefore obtain w_* directly from the expression for w by replacing in it δ_1 by

$$\delta_1' = \delta_1 - (1 + \nu). \tag{3.132}$$

In other words

$$w_*(r, \delta_1) = w(r, \delta_1'). \tag{3.133}$$

This procedure is called the method of equivalent rigidity.

2) This method can obviously be used also in the following way. Assume as known the deflections $w_*(r, \delta_1)$ of an annular plate ($R \leqslant r \leqslant R_1$) with reinforced inner edge. The relative flexural rigidity of the thin stiffening rib is δ_1. According to (3.133), we can directly obtain the deflections $w(r, \delta_1)$ of a solid plate under the same load having a concentric rib of equal rigidity δ_1:

$$w(r, \delta_1) = w_*(r, \delta_1') \quad (R \leqslant r \leqslant R_1), \tag{3.134}$$

where

$$\delta_1' = \delta_1 + 1 + \nu. \tag{3.135}$$

The deflections w_2 of the central part of the plate ($0 \leqslant r \leqslant R$) can be determined by solving the problem of bending of a circular plate with reinforced edge subjected to uniformly distributed bending moments transmitted to it from the annular part of the plate (cf. § 12, Chapter II). We obtain finally

$$w_2 = w_*(R, \delta_1') - \left[\frac{d^2 w_*(r, \delta_1')}{dr^2} + \frac{\nu}{r} \cdot \frac{dw_*(r, \delta_1')}{dr}\right]_{r=R} \cdot \frac{R^2 - r^2}{2(1+\nu+\delta_1)}. \tag{3.136}$$

Two examples of the application of this method are given below.

1) The annular plate is loaded uniformly along a circle (Figure 35). Replacing in (2.294) and (2.295) δ_1 by δ_1', given by (3.132), we obtain the solution for the plate shown in Figure 35:

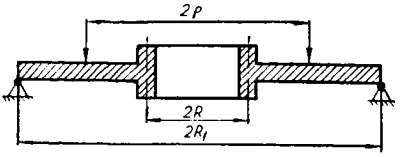


FIGURE 35.

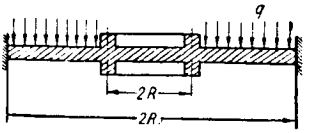


FIGURE 36.

for $R \leqslant r \leqslant \varrho$

$$w = w_1^0 - K_* \left[\frac{2(1-\nu)}{1+\nu} \left(1 - \frac{r^2}{R_1^2}\right) - 4 \ln \frac{r}{R_1} \right], \tag{3.137}$$

for $\varrho \leqslant r \leqslant R_1$

$$w = w_0^0 - K_* \left[\frac{2(1-\nu)}{1+\nu} \left(1 - \frac{r^2}{R_1^2}\right) - 4 \ln \frac{r}{R_1} \right], \tag{3.138}$$

where

$$K_* = \frac{PR^2 k \lambda_5(\delta_1, k)}{8\pi D} \left[\frac{1-\nu}{1+\nu} \left(1 - \frac{\varrho^2}{R_1^2}\right) + 2 \ln \frac{R_1}{\varrho} \right], \tag{3.139}$$

$$\lambda_5(\delta_1, k) = \lambda_1(\delta_1', k) = \frac{k}{4} \cdot \frac{(1+\nu)(\delta_1 - 1 - \nu)}{(1-\nu^2)(1-k^2) + \delta_1[k^2(1-\nu) + (1+\nu)]}. \tag{3.140}$$

The functions w_1^0 and w_0^0 are defined, respectively, by (2.289) and (2.290).

When $\varrho = R$, (3.138) reduces to an expression differing from (3.102) only by its sign. (Compare Figure 28 with Figure 35 for $\varrho = R$).

2) Circular plate with concentric rib partially loaded by a normal pressure (Figure 36). Applying the method of equivalent rigidity, we replace δ_1 by $\delta_1' = \delta_1 + 1 + \nu$ ((3.135)) in (3.127). As a result, we obtain for $R \leqslant r \leqslant R_1$:

$$w_* = \frac{qR^4}{64D[(\eta^2-1)\delta_1 + 2\eta^2]} \left\{ [(\eta^2-1)\delta_1 + 2\eta^2] \left(\frac{r^4}{R^4} - 8\frac{r^2}{R^2} \ln \frac{r}{R} \right) + \right.$$
$$+ \eta^2 [\eta^4 (2+\delta_1) - \eta^2 (8+3\delta_1) - 4\eta^2 \delta_1 \ln \eta + 4(2-\delta_1) \ln \eta - 2(2-\delta_1) +$$
$$+ 16\delta_1 \ln^2 \eta] - 4\eta^2 [(2+\delta_1) - \delta_1 (\eta^2 - 4 \ln \eta)] \ln \frac{r}{R} + 2[(2-\delta_1) +$$
$$\left. + (2+\delta_1)(2\eta^2 - \eta^4 + 4\eta^2 \ln \eta)] \frac{r^2}{R^2} \right\}. \tag{3.141}$$

The deflections for $0 \leqslant r \leqslant R$ are obtained by substituting (3.141) in (3.136):

$$w_2 = \frac{qR^4}{64D[(\eta^2-1)\delta_1 + 2\eta^2]} \left\{ \delta_1 [(\eta^2-1)^2 (\eta^2-3) + 4\eta^2 (4 \ln \eta + 1 - \eta^2) \ln \eta] + \right.$$
$$\left. + 2(\eta^2-1)(\eta^4 - 5\eta^2 - 2) + 24\eta^2 \ln \eta - 4\left(1 - \frac{r^2}{R^2}\right)(1 - \eta^4 + 4\eta^2 \ln \eta) \right\}. \tag{3.142}$$

§ 7. PLATE OF ARBITRARY SHAPE, REINFORCED AROUND A CIRCULAR HOLE OF SMALL RADIUS. SOLUTION BY FOURIER SERIES

We assume as known the deflections w^0 of a plate of arbitrary shape, having a small circular hole of radius R, located at a sufficient distance

from the edge ($5R \leqslant d$, where d is the least distance from the center of the hole to the plate edge).

We shall now determine the effect of a thin elastic rib, reinforcing the edge of the circular hole, on the state of stress of the plate, by finding the functions $\varphi(z)$ and $\psi(z)$ which characterize the additional deflections w^1 due to the stiffening rib at $r = R$. We shall use the solution obtained in § 4 of this chapter, assuming the plate to be unbounded. (The origin of coordinates is placed at the center of the hole.)

We represent the functions $\varphi(z)$ and $\psi(z)$ in the form

$$\varphi(z) = \sum_{n=0}^{\infty} \beta_{-n} \frac{R^n}{z^n}, \qquad \psi(z) = \sum_{n=1}^{\infty} \alpha_{-n} \frac{R^n}{z^n}, \tag{3.143}$$

hence, in the solution of § 4 we must set

$$\alpha_0 = \alpha_n = \beta_n = 0 \qquad \text{for } n \geqslant 1 \tag{3.144}$$

and discard the boundary condition (3.80). As a result, the system (3.87) to (3.95) becomes

$$\left.\begin{gathered}
\frac{3+\nu}{1-\nu}(\delta_1+\delta_2)\beta_0 + (\delta_1-\delta_2)\alpha_{-2} = 0 \\
\frac{3+\nu}{4}\left(\frac{1}{\delta_2}-\frac{1}{\delta_1}\right)\beta_0 + \left[1+\frac{1-\nu}{4}\left(\frac{1}{\delta_2}+\frac{1}{\delta_1}\right)\right]\alpha_{-2} = 2b_{-2} \\
\alpha_{-1}\left(1+\frac{1-\nu}{\delta_1}\right) = 2b_{-1} \\
-(n-1)\beta_{-(n-1)} + \alpha_{-(n+1)} - 2B_{-(n+1)} = 2b_{-(n+1)} \\
\beta_{-(n-1)} - 2\overline{B}_{n-1} = 2\overline{b}_{n-1}
\end{gathered}\right\}. \tag{3.145}$$

Solving this system yields

$$\left.\begin{gathered}
\alpha_{-1} = \frac{2b_{-1}\delta_1}{1-\nu+\delta_1} \\
\beta_{-(n-1)} = \frac{(n-1)\overline{b}_{n-1}}{\Delta_n}[2(n+1)\delta_1\delta_2 + (1- \\
-\nu)(\delta_1+\delta_2)] - \frac{(n+1)b_{-(n+1)}}{\Delta_n}(1-\nu)(\delta_1-\delta_2) \\
\alpha_{-(n+1)} = \frac{n-1}{\Delta_n}\overline{b}_{n-1}[2(n^2-1)\delta_1\delta_2 + (n-1)(1-\nu)\times \\
\times(\delta_1+\delta_2) - (3+\nu)(\delta_1-\delta_2)] + \frac{n+1}{\Delta_n}b_{-(n+1)}[2(n-1)\delta_1\delta_2 + \\
+(3+\nu)(\delta_1+\delta_2) - (n-1)(1-\nu)(\delta_1-\delta_2)]
\end{gathered}\right\}, \tag{3.146}$$

where

$$\Delta_n = (n^2-1)\delta_1\delta_2 + (1+\nu+2n)(\delta_1+\delta_2) + (1-\nu)(3+\nu) \quad (n \geqslant 1). \tag{3.147}$$

Here b_{n-1} and $b_{-(n+1)}$ are the Fourier coefficients of the function $\left(\frac{\partial w^0}{\partial t}\right)_{r=R}$ assumed to be known (cf. (3.81)).

From (3.146) we find the functions $\varphi(z)$ and $\psi(z)$ in (3.143). This solves the problem.

Bending in one direction of a rectangular plate

A rectangular unbounded plate, reinforced around a circular hole, is bent by applying moments $M_x = M$ distributed uniformly along two opposite edges (Figure 37).

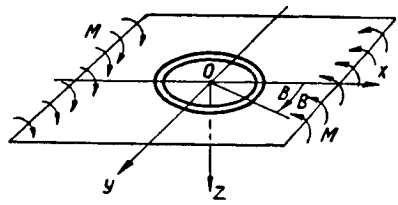


FIGURE 37.

Determining $\left(\frac{\partial w^0}{\partial t}\right)_{r=R}$ from the known solution /118/ for a nonreinforced plate with circular hole, we find

$$b_{-1} = \frac{MR}{4D(1-\nu^2)}, \qquad b_1 = \frac{MR}{2D(1-\nu)(3+\nu)}. \tag{3.148}$$

All the other coefficients $b_n (n = 0, \pm 2, \pm 3, \ldots)$ vanish. Substituting (3.148) in (3.146), we determine the coefficients of the functions $\varphi(z)$ and $\psi(z)$ in (3.143):

$$\alpha_{-1} = \frac{MR\delta_1}{2D(1-\nu^2)(1-\nu+\delta_1)},$$

$$\alpha_{-3} = \frac{MR}{2D(1-\nu)(3+\nu)\Delta_2}[6\delta_1\delta_2 - 2(1+\nu)\delta_1 + 4\delta_2],$$

$$\beta_{-1} = \frac{MR}{2D(1-\nu)(3+\nu)\Delta_2}[6\delta_1\delta_2 + (1-\nu)(\delta_1+\delta_2)],$$

$$\Delta_2 = (1-\nu)(3+\nu) + (5+\nu)(\delta_1+\delta_2) + 3\delta_1\delta_2.$$

The functions $\varphi^*(z)$ and $\psi^*(z)$, which describe the total deflection of the plate, are

$$\varphi^*(z) = -\frac{MR}{8D}\left(\frac{1}{1+\nu}\cdot\frac{z}{R} + \beta^*_{-1}\frac{R}{z}\right),$$

$$\psi^*(z) = -\frac{MR}{8D}\left(\frac{2}{1-\nu}\cdot\frac{z}{R} + \alpha^*_{-1}\frac{R}{z} + \alpha^*_{-3}\frac{R^3}{z^3}\right),$$

where

$$\left.\begin{aligned}
\alpha^*_{-1} &= 2(1+\nu-\delta_1)/(1+\nu)(1-\nu+\delta_1)\\
\beta^*_{-1} &= \frac{2}{(1-\nu)\Delta_2}[(1-\nu)^2+(1-\nu)(\delta_1+\delta_2)-3\delta_1\delta_2]\\
\alpha^*_{-3} &= \frac{2}{(1-\nu)\Delta_2}[(1-\nu)^2+(3-\nu)\delta_1-(1+\nu)\delta_2-3\delta_1\delta_2]
\end{aligned}\right\}. \qquad (3.149)$$

In this case we cannot obtain complete similarity between the state of strain of a plate with circular hole whose edge is reinforced by a rib of constant cross section, and that of a solid plate, since this would require that all three coefficients (3.149) vanish; this would yield three equations for determining the two unknown rigidities δ_1 and δ_2.

It is, however, possible to select such rigidities of the rib that the stress concentrations in the plate will be greatly reduced. Thus, for

$$\delta_1 = \delta_2 = 0.85, \qquad (3.150)$$

we obtain for $\nu=0.3$: $\alpha^*_{-1} = 0.4467$, $\alpha^*_{-3} = \beta^*_{-1} = -0.1033$.

The moments acting on the plate along $\Gamma(r = R)$ are then

$$M_\theta = M(0.5782 - 0.4509\cos 2\theta),$$
$$M_r = M(0.4218 + 0.5852\cos 2\theta),$$
$$H_{r\theta} = -0.4819 M \sin 2\theta.$$

The bending moment becomes maximum at $\theta = \pm 90°$ where

$$M_{\max} = 1.029 M.$$

TABLE 10

δ_2/δ_1	δ_1								$\frac{h_1}{b}$
	0	0.7	1	1.3	2	5	20	∞	
0.038	1.788	0.996	0.872	0.786	0.658	0.436	0.152	—0.626	6
0.081	1.788	1.005	0.879	0.789	0.651	0.387	0.006	—0.626	4
0.135	1.788	1.016	0.887	0.793	0.644	0.339	—0.096	—0.626	3
0.184	1.788	1.025	0.893	0.796	0.638	0.305	—0.152	—0.626	2.5
0.263	1.788	1.038	0.903	0.801	0.631	0.264	—0.210	—0.626	2
0.323	1.788	1.048	0.909	0.804	0.626	0.239	—0.239	—0.626	1.75
0.402	1.788	1.059	0.917	0.807	0.621	0.214	—0.290	—0.626	1.5
0.649	1.788	1.087	0.935	0.815	0.608	0.162	—0.313	—0.626	1
1.000	1.788	1.115	0.953	0.823	0.597	0.122	—0.343	—0.626	0.56

Thus, a rib whose relative rigidities are given by (3.150), prevents almost completely stress concentrations in the plate, and is practically optimum. Tables 10 and 11 give, respectively, values of M_θ/M for $\theta = 90°$ and of M_r/M for $\theta = 0$ at $r = R$ for different values of δ_1 and δ_2. The last columns of these tables contain the corresponding values of h_1/b (cf. Table 14 below) for a rib of rectangular cross section. We can see that for $0.7 \leqslant \delta_1 \leqslant 1.3$ a variation of δ_2 has little influence on the calculated stresses in the plate. In other words, the coefficient of stress concentration in the

plate depends mainly on the flexural rigidity of the rib, and far less on its torsional rigidity.

TABLE 11

δ_2/δ_1	δ_1								$\frac{h_1}{b}$
	0	0,7	1	1,3	2	5	20	∞	
0.038	0	0.202	0.261	0 309	0.397	0.643	1.338	3.626	6
0.081	0	0.255	0.334	0.401	0.531	0.918	1.885	3.626	4
0.135	0	0.317	0.418	0.505	0.677	1.183	2.267	3.626	3
0.184	0	0.368	0.487	0.590	0.791	1.372	2.480	3.626	2.5
0.263	0	0.444	0.586	0.709	0.947	1.605	2.697	3.626	2
0.323	0	0.496	0.653	0.788	1.048	1.741	2.806	3.626	1.75
0.402	0	0.559	0.731	0.846	1.161	1.884	2.907	3.626	1.5
0.649	0	0.720	0.926	1.100	1.420	2.175	3.083	3.626	1
1.000	0	0.885	1.117	1.307	1.648	2.396	3.196	3.626	0.56

Bending of a strip of finite width

A thin plate simply supported at the edges $y = \pm a$ and having the shape of a long strip of width $2a$, has a circular hole whose center lies on the axis of symmetry of the strip. The edge of the hole is reinforced by a thin elastic rib, and the plate is bent by applying moments $M_y = M$ uniformly distributed along the edges (Figure 38).

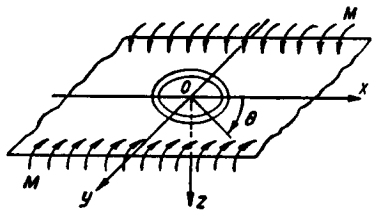


FIGURE 38.

Determining $\left(\frac{\partial w^0}{\partial t}\right)_{r=R}$ from the known solution /118/ for a nonreinforced plate with circular hole, we find

$$b_{-1} = \frac{MR}{2D(1-\nu)}, \quad b_1 = -\frac{MR}{D(3+\nu)}, \quad b_n = 0 \ (n=0, \pm 2, \pm 3, \ldots).$$

The functions $\varphi^*(z)$ and $\psi^*(z)$, which describe the total deflection of the plate, are

$$\varphi^*(z) = -\frac{MR}{4D}\left(\frac{z}{R} + \beta^*_{-1}\frac{R}{z}\right),$$

$$\psi^*(z) = -\frac{MR}{4D}\left(-2\frac{z}{R} + \alpha^*_{-1}\frac{R}{z} + \alpha^*_{-3}\frac{R^3}{z^3}\right),$$

where

$$\alpha^*_{-1} = 2(1 + \nu - \delta_1)/(1 - \nu + \delta_1),$$

$$\beta^*_{-1} = -\frac{2}{\Delta_2}[(1-\nu)^2 + (1-\nu)(\delta_1 + \delta_2) - 3\delta_1\delta_2],$$

$$\alpha^*_{-3} = -\frac{2}{\Delta_2}[(1-\nu)^2 + (3-\nu)\delta_1 - (1+\nu)\delta_2 - 3\delta_1\delta_2].$$

If we set

$$\delta_1 = \delta_2 = 0.95, \tag{3.151}$$

we obtain $\alpha^*_{-1} = 0.4242$, $\alpha^*_{-3} = \beta^*_{-1} = 0.1176$. The bending moment acting on the plate along $r=R$ becomes maximum at $\theta = \pm 90°$:

$$M_{\max} = 1.023\,M.$$

TABLE 12

δ_2/δ_1	δ_1								$\frac{h_1}{b}$
	0	0.7	1	1.3	2	5	20	∞	
0.038	1.852	1.067	0.939	0.850	0.718	0.500	0.243	—0.300	6
0.081	1.852	1.073	0.944	0.852	0.714	0.466	0.141	—0.300	4
0.135	1.852	1.080	0.949	0.855	0.709	0.433	0.070	—0.300	3
0.184	1.852	1.087	0.954	0.857	0.705	0.409	0.030	—0.300	2.5
0.263	1.852	1.096	0.960	0.861	0.700	0.380	—0.010	—0.300	2
0.323	1.852	1.103	0.965	0.863	0.696	0.363	—0.030	—0.300	1.75
0.402	1.852	1.110	0.970	0.865	0.693	0.345	—0.049	—0.300	1.5
0.649	1.852	1.130	0.983	0.871	0.683	0.308	—0.082	—0.300	1
1.000	1.852	1.150	0.995	0.876	0.676	0.281	—0.103	—0.300	0.56

TABLE 13

δ_2/δ_1	δ_1								$\frac{h_1}{b}$
	0	0.7	1	1.3	2	5	20	∞	
0.038	0	0.372	0.454	0.516	0.619	0.855	1.382	3.000	6
0.081	0	0.409	0.505	0.581	0.714	1.047	1.765	3.000	4
0.135	0	0.452	0.564	0.654	0.815	1.233	2.033	3.000	3
0.184	0	0.489	0.612	0.713	0.896	1.365	2.182	3.000	2.5
0.263	0	0.542	0.682	0.796	1.005	1.528	2.334	3.000	2
0.323	0	0.578	0.728	0.851	1.075	1.624	2.410	3.000	1 75
0.402	0	0.622	0.784	0.916	1.154	1.724	2.481	3.000	1.5
0.649	0	0.735	0.920	1.070	1.336	1.927	2.604	3.000	1
1.000	0	0.850	1.053	1.215	1.495	2.082	2.683	3.000	0.56

Thus, a stiffening rib whose relative rigidities are given by (3.151), prevents almost completely stress concentrations in the plate.

Tables 12 and 13 give, respectively, values of M_θ/M for $\theta = 0$ and of M_r/M for $\theta = 90°$ at $r = R$ for different values of δ_1 and δ_2. The last columns of these tables contain the corresponding values of $\frac{h_1}{b}$ (cf. Table 14 below) for a rib of rectangular cross section.

Tables 12 and 13 show that for values of δ_1 close to unity, a variation of δ_2 has little influence on the calculated stresses in the plate. The same is true for a variation of the ratio between the characteristic dimensions h_1 and b of the rib section.

§ 8. SOLUTION BY QUADRATURES FOR THE EXTERIOR OF THE CIRCLE

The edge Γ of a circular hole of radius R in an unbounded plane is reinforced by a thin elastic rib. The origin of coordinates is located at the center of the hole.

Let $z = \omega(\zeta) = R\zeta$. whence $t = R\sigma$, where $\sigma = e^{i\theta}$. The boundary conditions (3.1) to (3.2) then become

$$\left.\begin{aligned} -\varkappa\overline{\varphi(\sigma)} + \bar{\sigma}\varphi'(\sigma) + \psi(\sigma) &= f(\sigma) \\ \overline{\varphi(\sigma)} + \bar{\sigma}\varphi'(\sigma) + \psi(\sigma) &= g_*(\sigma) - g_0(\sigma) \end{aligned}\right\} \quad \text{on } \gamma, \tag{3.152}$$

where γ is the unit circle*,

$$\left.\begin{aligned} f(\sigma) &= -(\varkappa+1)\,\overline{g(R\sigma)}, \quad g_*(\sigma) = P(R\sigma) + \frac{C_3}{2i} \\ g_0(\sigma) &= \frac{1}{2}\left(\frac{\partial w^0}{\partial x} - i\frac{\partial w^0}{\partial y}\right)_\Gamma \end{aligned}\right\}. \tag{3.153}$$

We introduce the functions

$$\left.\begin{aligned} I^*(\sigma) &= -2iD(1-\nu)\sigma f(\sigma) \\ G(\sigma) &= 2iD(1-\nu)\sigma g_*(\sigma), \quad G_0(\sigma) = 2iD(1-\nu)\sigma g_0(\sigma) \end{aligned}\right\}. \tag{3.154}$$

The first two of these functions are unknown, while the third one is known. The function $g_*(\sigma)$, which describes the strains of the stiffening rib, is related to the function $I^*(\sigma)$, which depends on the load transmitted to this rib by the plate, by (3.6), which in this case becomes

$$-4\sigma\frac{dg_*(\sigma)}{Rd\theta} = \left(\frac{1}{C} + \frac{1}{A}\right)I^*(\sigma) + \left(\frac{1}{C} - \frac{1}{A}\right)\overline{I^*(\sigma)}. \tag{3.155}$$

The polar angle θ is measured clockwise from the x-axis, and therefore $Rd\theta = -ds$.

We subtract and add boundary conditions (3.152) after multiplying both sides by $2iD(1-\nu)\sigma$ and obtain on γ:

$$I^*(\sigma) + G(\sigma) - G_0(\sigma) = 8iD\sigma\,\overline{\varphi(\sigma)}, \tag{3.156}$$

$$\frac{1}{\varkappa}I^*(\sigma) - G(\sigma) + G_0(\sigma) = -\frac{8iD}{\varkappa}[\varphi'(\sigma) + \sigma\psi(\sigma)]. \tag{3.157}$$

* The functions $g(R\sigma)$ and $P(R\sigma)$ are defined by (3.3).

The right-hand side of (3.156) represents the boundary value of a function holomorphic inside γ and vanishing at $\zeta = 0$, while the right-hand side of (3.157) is the boundary value of a function holomorphic outside γ, including the point at infinity, where it acquires the unknown value $-\frac{8iD}{\varkappa} b_1 = \text{const}$. ($b_1$ is the coefficient of ζ^{-1} in the expansion of the function $\psi(\zeta)$; $\psi(\infty) = 0$).

We represent the functions (3.154) in the form

$$\left.\begin{aligned} &G(\sigma) = G_1(\sigma) + \Gamma + G_2(\sigma), \quad G_0(\sigma) = G_{01}(\sigma) + \Gamma_0 + G_{02}(\sigma) \\ &I^*(\sigma) = I_1(\sigma) + I_0 + I_2(\sigma) \end{aligned}\right\}, \tag{3.158}$$

where $G_1(\sigma)$, $G_{01}(\sigma)$, $I_1(\sigma)$ are, respectively, the boundary values of the functions $G_1(\zeta)$, $G_{01}(\zeta)$, $I_1(\zeta)$, holomorphic inside γ and vanishing at $\zeta=0$; $G_2(\sigma)$, $G_{02}(\sigma)$, $I_2(\sigma)$ are, respectively, the boundary values of the functions $G_2(\zeta)$, $G_{02}(\zeta)$, $I_2(\zeta)$ holomorphic outside γ and vanishing at infinity; Γ, Γ_0, I_0 are constants.

Multiplying both sides of (3.156) by the Cauchy kernel $\frac{1}{2\pi i}\cdot\frac{d\sigma}{\sigma-\zeta}$ and integrating along γ, first for $|\zeta|>1$ and then for $|\zeta|<1$, we obtain

$$\left.\begin{aligned} &I_2(\zeta) + G_2(\zeta) - G_{02}(\zeta) = 0 \\ &I_1(\zeta) + I_0 + G_1(\zeta) + \Gamma - G_{01}(\zeta) - \Gamma_0 = 8iD\zeta\overline{\varphi}\left(\frac{1}{\zeta}\right) \end{aligned}\right\}. \tag{3.159}$$

Similarly, we integrate (3.157) for $|\zeta|<1$, and obtain

$$\frac{1}{\varkappa}[I_1(\zeta) + I_0] - G_1(\zeta) - \Gamma + G_{01}(\zeta) + \Gamma_0 = -\frac{8iD}{\varkappa} b_1. \tag{3.160}$$

Setting $\zeta = 0$ in (3.160) and in the second equation (3.159), we obtain

$$-8iDb_1 + \varkappa\Gamma - I_0 = \varkappa\Gamma_0, \quad I_0 + \Gamma = \Gamma_0. \tag{3.161}$$

Let now $\zeta \to \sigma$ in (3.159) and (3.160). Taking (3.161) into account, we obtain

$$I_2(\sigma) = -G_2(\sigma) + G_{02}(\sigma), \quad I_1(\sigma) = \varkappa G_1(\sigma) - \varkappa G_{01}(\sigma). \tag{3.162}$$

Since

$$\sigma\frac{dg_*(\sigma)}{Rd\theta} = \frac{1}{R}\cdot\frac{d}{d\theta}[\sigma g_*(\sigma)] - \frac{i}{R}\sigma g_*(\sigma) = \frac{i}{R}\left\{\sigma\frac{d}{d\sigma}[\sigma g_*(\sigma)] - \right.$$

$$\left. - \sigma g_*(\sigma)\right\} = \frac{1}{2RD(1-\nu)}[\sigma G'(\sigma) - G(\sigma)],$$

we can rewrite (3.155) in the form

$$\sigma G'(\sigma) - G(\sigma) + \frac{1-\nu}{2\delta_1\delta_2}[(\delta_1+\delta_2)I^*(\sigma) + (\delta_1-\delta_2)\overline{I^*(\sigma)}] = 0 \text{ on } \gamma; \tag{3.163}$$

where

$$\delta_1 = \frac{A}{RD}, \quad \delta_2 = \frac{C}{RD}. \tag{3.164}$$

In the same way as we derived (3.161) and (3.162) from (3.156) and (3.157), we obtain from (3.163):

$$\left.\begin{aligned} \sigma G_1'(\sigma) - G_1(\sigma) + \frac{1-\nu}{2\delta_1\delta_2}[(\delta_1+\delta_2) I_1(\sigma) + (\delta_1-\delta_2)\,\overline{I_2(\sigma)}] = 0, \\ \sigma G_2'(\sigma) - G_2(\sigma) + \frac{1-\nu}{2\delta_1\delta_2}[(\delta_1+\delta_2) I_2(\sigma) + (\delta_1-\delta_2)\,\overline{I_1(\sigma)}] = 0, \end{aligned}\right\} \tag{3.165}$$

$$-\Gamma + \frac{1-\nu}{2\delta_1\delta_2}[(\delta_1+\delta_2) I_0 + (\delta_1-\delta_2)\bar{I}_0] = 0. \tag{3.166}$$

Taking (3.162) into account, and noting that

$$\sigma G_k'(\sigma) = -i\frac{dG_k}{d\theta} \quad (k=1,\ 2), \tag{3.167}$$

we obtain from (3.165) two ordinary differential equations with constant coefficients for determining the unknown functions $G_k(\sigma) = G_k(\theta) = G_k$ $(k = 1,\ 2)$:

$$\left.\begin{aligned} \frac{d\bar{G}_1}{d\theta} + a_{11}\bar{G}_1 + a_{12}G_2 = V_1 \\ \frac{dG_2}{d\theta} + a_{21}\bar{G}_1 + a_{22}G_2 = V_2 \end{aligned}\right\}, \tag{3.168}$$

where*

$$\left.\begin{aligned} a_{11} = i\left[1 - \frac{3+\nu}{2}\left(\frac{1}{\delta_2}+\frac{1}{\delta_1}\right)\right] \\ a_{12} = \frac{i(1-\nu)}{2}\left(\frac{1}{\delta_2}-\frac{1}{\delta_1}\right) \\ a_{22} = -i\left[1 + \frac{1-\nu}{2}\left(\frac{1}{\delta_2}+\frac{1}{\delta_1}\right)\right] \\ a_{21} = \frac{i(3+\nu)}{2}\left(\frac{1}{\delta_2}-\frac{1}{\delta_1}\right) \end{aligned}\right\} \tag{3.169}$$

$$\left.\begin{aligned} V_1 = V_1(\sigma) = \frac{i(1-\nu)}{2\delta_1\delta_2}[(\delta_1-\delta_2)G_{02} - (\delta_1+\delta_2)\varkappa\bar{G}_{01}] \\ V_2 = V_2(\sigma) = \frac{i(1-\nu)}{2\delta_1\delta_2}[(\delta_1-\delta_2)\varkappa\bar{G}_{01} - (\delta_1+\delta_2)G_{02}] \end{aligned}\right\}. \tag{3.170}$$

* It can be shown on the basis of (3.153), (3.154), (3.158), and (3.170) that the functions V_1 and V_2 are single-valued on γ.

We also obtain from (3.161) and (3.166):

$$\left.\begin{aligned} &\operatorname{Re} I_0 = \frac{\delta_2}{1-\nu+\delta_2}\operatorname{Re}\Gamma_0, \quad \operatorname{Im} I_0 = \frac{\delta_1}{1-\nu+\delta_1}\operatorname{Im}\Gamma_0 \\ &\operatorname{Re}\Gamma = \frac{1-\nu}{1-\nu+\delta_2}\operatorname{Re}\Gamma_0, \quad \operatorname{Im}\Gamma = \frac{1-\nu}{1-\nu+\delta_1}\operatorname{Im}\Gamma_0 \\ &b_1 = \frac{1}{2D(1-\nu)}\left(\frac{i\delta_2}{1-\nu+\delta_2}\operatorname{Re}\Gamma_0 - \frac{\delta_1}{1-\nu+\delta_1}\operatorname{Im}\Gamma_0\right) \end{aligned}\right\}. \tag{3.171}$$

To solve system (3.168), we form the characteristic equation

$$\lambda^2 + (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0, \tag{3.172}$$

from which we find

$$\lambda_{1,2} = i\mu_{1,2} = i\left\{\left(\frac{1}{\delta_2}+\frac{1}{\delta_1}\right) \pm \right.$$
$$\left. \pm \sqrt{\left[1 - \frac{1+\nu}{2}\left(\frac{1}{\delta_2}+\frac{1}{\delta_1}\right)\right]^2 + \frac{(1-\nu)(3+\nu)}{4}\left(\frac{1}{\delta_2}-\frac{1}{\delta_1}\right)^2}\right\}. \tag{3.173}$$

Here μ_1 and μ_2 are always real and different.

The general solution of system (3.168) is (cf. for instance Stepanov/133/)

$$\left.\begin{aligned} \bar{G}_1 &= C_1(a_{22}+i\mu_1)e^{i\mu_1\theta} + C_2(a_{22}+i\mu_2)e^{i\mu_2\theta} + \bar{Y}_1 \\ G_2 &= -C_1 a_{21}e^{i\mu_1\theta} - C_2 a_{21}e^{i\mu_2\theta} + Y_2 \end{aligned}\right\}, \tag{3.174}$$

where C_1, C_2 are still undetermined constants, and $\bar{Y}_1$, Y_2 are particular solutions, single-valued on γ, of system (3.168), obtained by varying the constants:

$$\left.\begin{aligned} \bar{Y}_1 &= \frac{ie^{i\mu_1\theta}}{\mu_2-\mu_1}\int[(a_{22}+i\mu_1)V_1 - a_{12}V_2]e^{-i\mu_1\theta}d\theta - \\ &- \frac{ie^{i\mu_2\theta}}{\mu_2-\mu_1}\int[(a_{22}+i\mu_2)V_1 - a_{12}V_2]e^{-i\mu_2\theta}d\theta \\ Y_2 &= \frac{ie^{i\mu_1\theta}}{\mu_2-\mu_1}\int[(a_{11}+i\mu_1)V_2 - a_{21}V_1]e^{-i\mu_1\theta}d\theta - \\ &- \frac{ie^{i\mu_2\theta}}{\mu_2-\mu_1}\int[(a_{11}+i\mu_2)V_2 - a_{21}V_1]e^{-i\mu_2\theta}d\theta \end{aligned}\right\}. \tag{3.175}$$

For the functions G_1 and G_2 in (3.174) to be single-valued, it is necessary that

$$C_1 = C_2 = 0.$$

and therefore

$$\overline{G_1(\sigma)} = \bar{Y}_1, \quad G_2(\sigma) = Y_2. \tag{3.176}$$

The functions

$$\left.\begin{aligned}
\bar{G}_1\left(\frac{1}{\zeta}\right) &= \frac{\zeta^{\mu_1}}{\mu_2 - \mu_1}\int [(a_{22} + i\mu_1) V_1(\zeta) - \\
&- a_{12}V_2(\zeta)]\,\zeta^{-(1+\mu_1)}d\zeta - \frac{\zeta^{\mu_2}}{\mu_2 - \mu_1}\int [(a_{22} + i\mu_2) V_1(\zeta) - \\
&- a_{12}V_2(\zeta)]\,\zeta^{-(1+\mu_2)}d\zeta \\
G_2(\zeta) &= \frac{\zeta^{\mu_1}}{\mu_2 - \mu_1}\int [(a_{11} + i\mu_1) V_2(\zeta) - a_{21}V_1(\zeta)] \times \\
&\times \zeta^{-(1+\mu_1)}d\zeta - \frac{\zeta^{\mu_2}}{\mu_2 - \mu_1}\int [(a_{11} + i\mu_2) V_2(\zeta) - \\
&- a_{21}V_1(\zeta)]\,\zeta^{-(1+\mu_2)}d\zeta
\end{aligned}\right\} \tag{3.177}$$

are holomorphic outside γ.

The unknown functions $\varphi(\zeta)$ and $\psi(\zeta)$ are directly expressed through the functions (3.177). In fact, solving the boundary-value problem (3.152), taking (3.158) [3.161 and 3.162] into account, we obtain

$$\left.\begin{aligned}
\varphi(\zeta) &= \frac{i}{2D(1-\nu)}\left[\zeta\bar{G}_1\left(\frac{1}{\zeta}\right) - \bar{G}_1(0) - \zeta\bar{G}_{01}\left(\frac{1}{\zeta}\right) + \bar{G}_{01}(0)\right], \\
\psi(\zeta) &= \frac{i}{2D(1-\nu)\zeta}\left[-\Gamma + \Gamma_0 - G_2(\zeta) + G_{02}(\zeta) - \right. \\
&\left. - \bar{G}_1\left(\frac{1}{\zeta}\right) + \bar{G}_{01}\left(\frac{1}{\zeta}\right) - \zeta\frac{d}{d\zeta}\bar{G}_1\left(\frac{1}{\zeta}\right) + \zeta\frac{d}{d\zeta}\bar{G}_{01}\left(\frac{1}{\zeta}\right)\right]
\end{aligned}\right\} \tag{3.178}$$

This solves the problem.

The advantages of this solution over that by Fourier series are most evident when the plate is loaded by concentrated forces or moments.

Several simple examples are given to illustrate the procedure for solving problems by this method.

Pure bending of a plate reinforced around a circular hole

An unbounded plate, reinforced around a circular hole of radius , is bent by applying moments $M_x^{(\infty)} = M_y^{(\infty)} = M$ at infinity.

We have in this case for a nonreinforced plate /118/

$$w^0 = -\frac{M}{2D}\left[\frac{1}{1+\nu}(x^2 + y^2) + \frac{R^2}{1-\nu}\ln(x^2 + y^2)\right].$$

$$\varphi^0(\zeta) = -\frac{MR}{4D}\cdot\frac{1}{1+\nu}\cdot\zeta, \quad \psi^0(\zeta) = -\frac{MR}{2D}\cdot\frac{1}{1-\nu}\cdot\frac{1}{\zeta}. \tag{3.179}$$

The functions correspond to this basic deflection. From (3.153), (3.154), (3.158), (3.170), (3.171), (3.175), and (3.176), we obtain

$$g_0(\sigma) = -MR/D(1-\nu^2)\sigma, \quad G_0(\sigma) = -2iMR/(1+\nu),$$

$$G_{01}(\sigma) = G_{02}(\sigma) = 0, \quad \Gamma_0 = -2iMR/(1+\nu),$$

$$V_1 = V_2 = 0, \quad \Gamma = -2iMR(1-\nu)/(1+\nu)(1-\nu+\delta_1),$$

$$b_1 = MR\delta_1/D(1-\nu^2)(1-\nu+\delta_1), \quad \bar{Y}_1 = Y_2 = \overline{G_1(\sigma)} = G_2(\sigma) = 0.$$

Hence, by (3.178):

$$\varphi(\zeta) = 0, \quad \psi(\zeta) = \frac{MR}{D(1-\nu^2)}\cdot\frac{\delta_1}{1-\nu+\delta_1}\cdot\frac{1}{\zeta}. \tag{3.180}$$

The functions (3.180) describe the additional stresses due to the stiffening rib. The total stresses are described by functions obtained by adding (3.179) and (3.180):

$$\varphi_1(\zeta) = \varphi^0(\zeta) + \varphi(\zeta) = -\frac{MR}{4D}\cdot\frac{1}{1+\nu}\zeta,$$

$$\psi_1(\zeta) = \psi^0(\zeta) + \psi(\zeta) = \frac{MR}{2D}\cdot\frac{\delta_1 - 1 - \nu}{(1+\nu)(1-\nu+\delta_1)}\cdot\frac{1}{\zeta}.$$

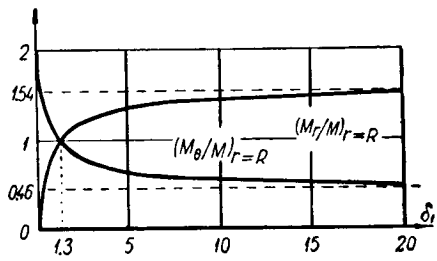


FIGURE 39

The bending moments in the plate are

$$\left.\begin{aligned} M_r &= M\left(1 - \frac{1-\nu}{1+\nu}\cdot\frac{1+\nu-\delta_1}{1-\nu+\delta_1}\cdot\frac{R^2}{r^2}\right) \\ M_\theta &= M\left(1 + \frac{1-\nu}{1+\nu}\cdot\frac{1+\nu-\delta_1}{1-\nu+\delta_1}\cdot\frac{R^2}{r^2}\right) \end{aligned}\right\}. \tag{3.181}$$

Figure 39 shows $\left(\frac{M_r}{M}\right)_{r=R}$ and $\left(\frac{M_\theta}{M}\right)_{r=R}$ as functions of δ_1 for $\nu = 0.3$.

It follows from (3.181) that for $\delta_1 = 1 + \nu$ throughout the plate $M_r = M_\theta = M$, i. e., a stiffening rib of relative rigidity $\delta_1 = 1 + \nu$ prevents completely stress concentrations in the plate.

If the stiffening rib has a rectangular cross section, then $A = E_1 I_n = E_1 b h_1^3/12$ and

$$\delta_1 = 1 + \nu = \frac{A}{RD} = \frac{E_1}{E} \cdot \frac{b}{R} \left(\frac{h_1}{h}\right)^3 (1 - \nu^2). \tag{3.182}$$

For a given plate material (E, ν) we can select arbitrarily (within the limits of applicability of the above solution for thin beams /131/) two of the three dimensionless factors in the right-hand side of (3.182), and determine the third from this equation.

Torsion of an unbounded plate reinforced around a circular hole

For a nonreinforced plate with circular hole under pure torsion at infinity (Figure 40), we have /118/

$$\left.\begin{aligned} w^0 &= \frac{HR^2}{2D(1-\nu)} \left| \frac{1-\nu}{3+\nu} \left(\frac{R^2}{r^2} - 2\right) - \frac{r^2}{R^2} \right| \sin 2\theta \\ \varphi^0(\zeta) &= -\frac{iHR}{2D(1-\nu)} \cdot \frac{1-\nu}{3+\nu} \cdot \frac{1}{\zeta} \\ \psi^0(\zeta) &= \frac{iHR}{2D(1-\nu)} \left[\zeta - \frac{1-\nu}{3+\nu} \cdot \frac{1}{\zeta^3} \right] \end{aligned}\right\}. \tag{3.183}$$

From (3.153), (3.154), (3.158), (3.170), (3.171), (3.175), (3.176), and (3.178), we find

$$\left.\begin{aligned} g_0(\sigma) &= \frac{2iHR}{D(1-\nu)(3+\nu)} \sigma, \quad G_0(\sigma) = -\frac{4HR}{3+\nu} \sigma^2 \\ G_{01}(\sigma) &= -\frac{4HR}{3+\nu} \sigma^2, \quad G_{02}(\sigma) = \Gamma_0 = 0 \\ V_1(\zeta) &= \frac{2iHR}{\zeta^2} \left(\frac{1}{\delta_1} + \frac{1}{\delta_2}\right) \\ V_2(\zeta) &= \frac{2iHR}{\zeta^2} \left(\frac{1}{\delta_1} - \frac{1}{\delta_2}\right), \quad \Gamma = b_1 = 0 \\ \varphi(\zeta) &= \frac{iHR}{D(1-\nu)(3+\nu)} \cdot \frac{(1-\nu)(\delta_1+\delta_2) + 6\delta_1\delta_2}{(1-\nu)(3+\nu) + (5+\nu)(\delta_1+\delta_2) + 3\delta_1\delta_2} \cdot \frac{1}{\zeta} \\ \psi(\zeta) &= \frac{iHR}{2D(3+\nu)(1-\nu)} \cdot \frac{8\delta_2 - 4(1+\nu)\delta_1 + 12\delta_1\delta_2}{(1-\nu)(3+\nu) + (5+\nu)(\delta_1+\delta_2) + 3\delta_1\delta_2} \cdot \frac{1}{\zeta^3} \end{aligned}\right\}. \tag{3.184}$$

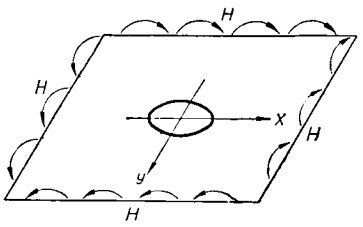


FIGURE 40.

Addition of (3.183) and (3.184) yields

$$\left.\begin{aligned}\varphi_1(\zeta) &= -\frac{iHR}{2D(1-\nu)}\times\\ &\times\frac{(1-\nu)^2+(1-\nu)(\delta_1+\delta_2)-3\delta_1\delta_2}{(1-\nu)(3+\nu)+(5+\nu)(\delta_1+\delta_2)+3\delta_1\delta_2}\cdot\frac{1}{\zeta}\\ \psi_1(\zeta) &= \frac{iHR}{2D(1-\nu)}\times\\ &\times\left[\zeta-\frac{1}{\zeta^3}\cdot\frac{(1-\nu)^2+(3-\nu)\delta_1-(1+\nu)\delta_2-3\delta_1\delta_2}{(1-\nu)(3+\nu)+(5+\nu)(\delta_1+\delta_2)+3\delta_1\delta_2}\right]\end{aligned}\right\}. \qquad (3.185)$$

For complete elimination of stress concentrations in the plate, it obviously suffices that

$$\left.\begin{aligned}(1-\nu)^2+(1-\nu)(\delta_1+\delta_2)-3\delta_1\delta_2=0\\ (1-\nu)^2+(3-\nu)\delta_1-(1+\nu)\delta_2-3\delta_1\delta_2=0\end{aligned}\right\}. \qquad (3.186)$$

Solution of system (3.186) yields

$$\delta_1=\delta_2=1-\nu. \qquad (3.187)$$

A rib, whose relative rigidities are given by (3.187), causes the calculated stresses in the plate to be minimum. Such a rib, and the corresponding rigidities δ_1 and δ_2, will be called optimum.

An optimum rib can be designed as follows, after the optimum rigidities have been found: first, the shape of the cross section is selected. Next, the magnitudes $\frac{\delta_2}{\delta_1}=\frac{I_\tau}{2(1+\nu)I_n}$ ((2.32)) are tabulated for the chosen shape as a function of the ratio of the two characteristic dimensions h_1 and b of the cross section. From this table we then find the value of h_1/b corresponding to the ratio of optimum rigidities $\delta_2^{opt}/\delta_1^{opt}$. Finally, having selected the material for the rib, we determine the absolute dimensions of its section from

$$\delta_1=\frac{E_1 I_n}{RD}. \qquad (3.188)$$

This procedure will be illustrated by selecting the cross section for case (3.187). Let the stiffening rib have a rectangular cross section of

height h_1 and width b. Writing

$$I_n = \varepsilon_1 b^4, \quad I_\tau = \varepsilon_k b^4, \tag{3.189}$$

we obtain Table 14 for $\nu=0.3$ /23/.

TABLE 14

$\frac{h_1}{b}$	0.5	0.6	1.0	1.1	1.5
ε_k	0.0287	0.0451	0.140	0.169	0.294
ε_1	0.0104	0.0180	0.083	0.111	0.281
δ_2/δ_1	1.059	0.965	0.649	0.588	0.402

In our case, $\frac{\delta_2^{\text{отг}}}{\delta_1^{\text{отг}}} = 1$ by (3.187). The corresponding value of h_1/b in Table 14 is approximately 0.56. Replacing by R by R_2 (our method is based on the assumption that this is permissible), we can represent (3.188) in the form

$$\delta_1 = \frac{E_1}{E} \cdot \frac{bh_1^3}{R_2 h^3}(1-\nu^2) = \frac{E_1}{E}\left(\frac{b}{h}\right)^4 \cdot \frac{(0.56)^3}{\varrho}(1-\nu^2) = 1-\nu, \tag{3.190}$$

where R_2 is the radius of the hole in the plate, and $\varrho = \frac{R_2}{h}$ is known.

Let plate and rib consist of the same material. We find from (3.190) for $E = E_1$: $b = 1{,}44h\sqrt[4]{\varrho}$. Knowing ϱ, we obtain first b, and then h_1 from $h_1 = 0.56b = 0.81h\sqrt[4]{\varrho}$.

The dimensions of the rib are thus expressed through the known thickness h of the plate.

§ 9. SOLUTION BY QUADRATURES FOR A CIRCLE

A circular plate is reinforced along the edge $r = R$ by a thin elastic rib. The origin of the xy-coordinates is at the center of the plate. The boundary conditions on γ are by (3.156) and (3.157):

$$I^*(\sigma) + G(\sigma) - G_0(\sigma) = 8iD\sigma\overline{\varphi(\sigma)}, \tag{3.191}$$

$$\frac{1}{\varkappa} I^*(\sigma) - G(\sigma) + G_0(\sigma) = -\frac{8iD}{\varkappa}[\varphi'(\sigma) + \sigma\psi(\sigma)]. \tag{3.192}$$

The same symbols are used here as in § 8. The right-hand side of (3.191) represents the boundary value of the function $\zeta\overline{\varphi}\left(\frac{1}{\zeta}\right)$, holomorphic outside γ, including the point at infinity, where it acquires the value $8iD\overline{a_1} = $ = const. (a_1 is the coefficient of ζ in the expansion of $\varphi(\zeta)$; it is assumed that $\varphi(0) = 0$). The right-hand side of (3.192) is the boundary value of the function $\frac{8iD}{\varkappa}[\varphi'(\zeta) + \zeta\psi(\zeta)]$, holomorphic inside γ and vanishing at $\zeta = 0$.

We again rewrite (3.154) in the form (3.158)

$$\left.\begin{aligned} G(\sigma) &= G_1(\sigma) + \Gamma + G_2(\sigma), \quad G_0(\sigma) = G_{01}(\sigma) + \Gamma_0 + G_{02}(\sigma) \\ I^*(\sigma) &= I_1(\sigma) + I_0 + I_2(\sigma) \end{aligned}\right\}. \tag{3.193}$$

Here $G_1(\sigma)$, $G_{01}(\sigma)$ and $I_1(\sigma)$ are, respectively, the boundary values of the functions $G_1(\zeta)$, $G_{01}(\zeta)$ and $I_1(\zeta)$, holomorphic outside γ and vanishing at infinity; $G_2(\sigma), G_{02}(\sigma)$ and $I_2(\sigma)$ are the boundary values of the functions $G_2(\zeta)$, $G_{02}(\zeta)$ and $I_2(\zeta)$, holomorphic inside γ and vanishing at $\zeta = 0$; Γ, Γ_0 and I_0 are constants.

Multiplying both sides of (3.191) and (3.192) by $\frac{1}{2\pi i}\cdot\frac{d\sigma}{\sigma-\zeta}$ and integrating, along γ, the first expression obtained for $|\zeta|<1$, and the second both for $|\zeta|<1$ and $|\zeta|>1$, we obtain

$$I_2(\zeta) + I_0 + G_2(\zeta) + \Gamma - G_{02}(\zeta) - \Gamma_0 = 8iD\bar{a}_1, \tag{3.194}$$

$$-\frac{1}{\varkappa} I_1(\zeta) + G_1(\zeta) - G_{01}(\zeta) = 0, \tag{3.195}$$

$$\frac{1}{\varkappa}[I_2(\zeta) + I_0] - G_2(\zeta) - \Gamma + G_{02}(\zeta) + \Gamma_0 = -\frac{8iD}{\varkappa}[\varphi'(\zeta) + \zeta\psi(\zeta)]. \tag{3.196}$$

Setting $\zeta = 0$ in (3.194) and (3.196), we obtain

$$I_0 + \Gamma - \Gamma_0 = 8iD\bar{a}_1, \quad I_0 - \varkappa\Gamma + \varkappa\Gamma_0 = -8iDa_1. \tag{3.197}$$

Letting $\zeta \to \sigma$ in (3.194) and (3.195) and taking (3.197) into account, we find

$$I_2(\sigma) = -G_2(\sigma) + G_{02}(\sigma), \quad I_1(\sigma) = \varkappa G_1(\sigma) - \varkappa G_{01}(\sigma). \tag{3.198}$$

This result is identical with (3.162). Furthermore,

$$4\sigma\frac{dg_*(\sigma)}{Rd\theta} = \left(\frac{1}{C} + \frac{1}{A}\right) I^*(\sigma) + \left(\frac{1}{C} - \frac{1}{A}\right)\overline{I^*(\sigma)}, \tag{3.199}$$

which differs from (3.155) by the sign on the left-hand side, because γ is described in the opposite sense.

Proceeding as in § 8, we reduce (3.199) to the following system of equations, similar to (3.165) and (3.166):

$$\left.\begin{aligned} -\sigma G_1'(\sigma) + G_1(\sigma) + \frac{1-\nu}{2\delta_1\delta_2}\left[(\delta_1+\delta_2) I_1(\sigma) + (\delta_1-\delta_2)\overline{I_2(\sigma)}\right] = 0 \\ -\sigma G_2'(\sigma) + G_2(\sigma) + \frac{1-\nu}{2\delta_1\delta_2}\left[(\delta_1+\delta_2) I_2(\sigma) + (\delta_1-\delta_2)\overline{I_1(\sigma)}\right] = 0 \end{aligned}\right\}, \tag{3.200}$$

$$\Gamma + \frac{1-\nu}{2\delta_1\delta_2}[(\delta_1+\delta_2) I_0 + (\delta_1-\delta_2)\bar{I}_0] = 0, \tag{3.201}$$

where δ_1 and δ_2 are defined by (3.164).

From (3.197) and (3.201) we obtain

$$\left.\begin{aligned} \operatorname{Im} I_0 &= -\frac{(1+\nu)\delta_1}{(1-\nu)(1+\nu+\delta_1)}\operatorname{Im}\Gamma_0, \quad \operatorname{Re} I_0 = -\frac{\delta_2}{1-\nu}\operatorname{Re}\Gamma_0 \\ \operatorname{Im}\Gamma &= \frac{1+\nu}{1+\nu+\delta_1}\operatorname{Im}\Gamma_0, \quad \operatorname{Re}\Gamma = \operatorname{Re}\Gamma_0 \\ a_1 &= -\frac{1}{8(1-\nu)D}\left(i\delta_2 \operatorname{Re}\Gamma_0 + \frac{2\delta_1}{1+\nu+\delta_1}\operatorname{Im}\Gamma_0\right) \end{aligned}\right\}. \tag{3.202}$$

We determine the functions G_1 and G_2 by substituting (3.198) in (3.200); using (3.167), we obtain a system of ordinary differential equations coinciding in form with system (3.168):

$$\left.\begin{aligned} \frac{d\bar{G}_1}{d\theta} + a_{11}\bar{G}_1 + a_{12}G_2 &= V_1, \\ \frac{dG_2}{d\theta} + a_{21}\bar{G}_1 + a_{22}G_2 &= V_2, \end{aligned}\right\} \tag{3.203}$$

where

$$\left.\begin{aligned} a_{11} &= i\left[1 + \frac{3+\nu}{2}\left(\frac{1}{\delta_2}+\frac{1}{\delta_1}\right)\right] \\ a_{12} &= \frac{i(1-\nu)}{2}\left(\frac{1}{\delta_1}-\frac{1}{\delta_2}\right) \\ a_{22} &= -i\left[1 - \frac{1-\nu}{2}\left(\frac{1}{\delta_2}+\frac{1}{\delta_1}\right)\right] \\ a_{21} &= \frac{i(3+\nu)}{2}\left(\frac{1}{\delta_1}-\frac{1}{\delta_2}\right) \end{aligned}\right\}, \tag{3.204}$$

$$\left.\begin{aligned} V_1 = V_1(\sigma) &= -\frac{i(1-\nu)}{2\delta_1\delta_2}\left[(\delta_1-\delta_2)G_{02} - (\delta_1+\delta_2)\varkappa\bar{G}_{01}\right] \\ V_2 = V_2(\sigma) &= -\frac{i(1-\nu)}{2\delta_1\delta_2}\left[(\delta_1-\delta_2)\varkappa\bar{G}_{01} - (\delta_1+\delta_2)G_{02}\right] \end{aligned}\right\}. \tag{3.205}$$

The characteristic equation of system (3.203) has the form of (3.172); its roots are

$$\lambda_{1,2} = i\mu_{1,2} = i\left\{-\left(\frac{1}{\delta_2}+\frac{1}{\delta_1}\right) \pm \right.$$
$$\left. \pm \sqrt{\left[1 + \frac{1+\nu}{2}\left(\frac{1}{\delta_2}+\frac{1}{\delta_1}\right)\right]^2 + \frac{(1-\nu)(3+\nu)}{4}\left(\frac{1}{\delta_2}-\frac{1}{\delta_1}\right)^2}\right\}, \tag{3.206}$$

where μ_1 and μ_2 are different and real. The single-valued solution of system (3.203), obtained similarly as in the preceding section, is

$$\bar{G}_1(\sigma) = \bar{Y}_1, \quad G_2(\sigma) = Y_2, \tag{3.207}$$

where $\overline{Y}_1$ and Y_2 are defined by (3.175) and (3.204) to (3.206). The functions $G_2(\sigma) = Y_2$ and $\overline{G}_1\left(\frac{1}{\sigma}\right) = \overline{Y}_1$, are, respectively, the boundary values of the functions $G_2(\zeta)$ and $\overline{G}_1\left(\frac{1}{\zeta}\right)$, holomorphic inside the unit circle, and are found by substituting ζ for σ. They have the form of (3.177), but with μ_j and a_{jk} $(j, k = 1, 2)$ defined by (3.206) and (3.204), and $V_1(\zeta)$ and $V_2(\zeta)$ obtained from (3.205) by replacing σ by ζ.

We can express the unknown functions $\varphi(\zeta)$ and $\psi(\zeta)$ through functions (3.177) by solving the second boundary-value problem (3.152), which, on the basis of (3.193), can be written as follows:

$$\overline{\varphi(\sigma)} + \overline{\sigma}\varphi'(\sigma) + \psi(\sigma) = \overline{F(\sigma)}. \tag{3.208}$$

where

$$\overline{F(\sigma)} = \frac{1}{2iD(1-\nu)} \cdot \frac{1}{\sigma} [G_1(\sigma) + \Gamma + G_2(\sigma) - G_{01}(\sigma) - \Gamma_0 - G_{02}(\sigma)]. \tag{3.209}$$

The solution of (3.208) for a circle, with $\varphi(0) = 0$, is (/82/, § 80):

$$\left.\begin{aligned} \varphi(\zeta) &= -\overline{a}_1\zeta - 2\overline{a}_2 - \overline{\psi(0)} + \frac{1}{2\pi i}\int_\gamma \frac{F(\sigma)\,d\sigma}{\sigma - \zeta} \\ \psi(\zeta) &= \frac{1}{2\pi i}\int_\gamma \frac{\overline{F}\,d\sigma}{\sigma - \zeta} - \frac{1}{\zeta}\varphi'(\zeta) + \frac{a_1}{\zeta} - 2a_2 \end{aligned}\right\}, \tag{3.210}$$

where

$$a_1 = \varphi'(0), \quad 2a_2 = \varphi''(0). \tag{3.211}$$

Differentiating $\varphi(\zeta)$ in (3.210) with respect to ζ and setting $\zeta = 0$, we obtain Re a_1:

$$a_1 + \overline{a}_1 = \frac{1}{2\pi i}\int_\gamma F(\sigma)\frac{d\sigma}{\sigma^2},$$

whence

$$a_1 = \frac{1}{4\pi i}\int_\gamma F(\sigma)\frac{d\sigma}{\sigma^2} + iC_*, \quad (C_* = \overline{C}_*). \tag{3.212}$$

Differentiating $\varphi(\zeta)$ twice with respect to ζ and setting $\zeta = 0$, we obtain

$$a_2 = \frac{1}{2\pi i}\int_\gamma F\frac{d\sigma}{\sigma^2} \quad \text{or} \quad \overline{a}_2 = \frac{1}{2\pi i}\int_\gamma \overline{F(\sigma)}\,\sigma d\sigma. \tag{3.213}$$

We determine the constant $\psi(0)$ in (3.210) by setting $\zeta = 0$ in the expression for $\psi(\zeta)$:

$$\psi(0) = \frac{1}{2\pi i}\int_\gamma \frac{\overline{F}d\sigma}{\sigma} - 2a_2 \quad \text{or} \quad \overline{\psi(0)} = \frac{1}{2\pi i}\int_\gamma \frac{F d\sigma}{\sigma} - 2\overline{a}_2. \tag{3.214}$$

Using (3.212) to (3.214), we can now rewrite (3.210) in the form

$$\left.\begin{aligned}\varphi(\zeta) &= -\frac{\zeta}{4\pi i}\int_\gamma \overline{F(\sigma)}d\sigma + iC_*\zeta - \\ &- \frac{1}{2\pi i}\int_\gamma \frac{Fd\sigma}{\sigma} + \frac{1}{2\pi i}\int_\gamma \frac{F(\sigma)d\sigma}{\sigma-\zeta} \\ \psi(\zeta) &= \frac{1}{2\pi i}\int_\gamma \frac{\overline{F(\sigma)}d\sigma}{\sigma-\zeta} + \frac{1}{\zeta}\left[\frac{1}{2\pi i}\int_\gamma F(\sigma)\frac{d\sigma}{\sigma^2} - \right. \\ &\left. - \frac{1}{2\pi i}\int_\gamma \frac{F'(\sigma)d\sigma}{\sigma-\zeta}\right] - \frac{1}{\pi i}\int_\gamma F(\sigma)\frac{d\sigma}{\sigma^3}\end{aligned}\right\}. \tag{3.215}$$

Substituting for $\overline{F(\sigma)}$ from (3.209), and setting

$$F(\sigma) = \frac{-\sigma}{2iD(1-\nu)}\left[\bar{G}_1\left(\frac{1}{\sigma}\right) + \bar{\Gamma} + \bar{G}_2\left(\frac{1}{\sigma}\right) - \bar{G}_{01}\left(\frac{1}{\sigma}\right) - \Gamma_0 - \bar{G}_{02}\left(\frac{1}{\sigma}\right)\right] \tag{3.216}$$

we obtain

$$\left.\begin{aligned}\varphi(\zeta) &= \frac{i}{2D(1-\nu)}\left[\frac{1}{2}(\Gamma_0-\Gamma)\zeta + \right. \\ &\left. + \zeta\bar{G}_1\left(\frac{1}{\zeta}\right) - \zeta\bar{G}_{01}\left(\frac{1}{\zeta}\right)\right] \\ \psi(\zeta) &= \frac{i}{2D(1-\nu)}\left[\frac{1}{\zeta}G_{02}(\zeta) - \frac{1}{\zeta}G_2(\zeta)\right] - \frac{1}{\zeta}\varphi'(\zeta) + \\ &+ \frac{1}{\zeta}\cdot\frac{i(\bar{\Gamma}-\bar{\Gamma}_0)}{4D(1-\nu)} + \frac{i}{D(1-\nu)}\left[\frac{d}{d\zeta}\bar{G}_{01}\left(\frac{1}{\zeta}\right) - \frac{d}{d\zeta}\bar{G}_1\left(\frac{1}{\zeta}\right)\right]_{\zeta=0}\end{aligned}\right\} \tag{3.217}$$

We have discarded the term $iC_*\zeta$ in $\varphi(\zeta)$, since it does not affect the deflections and stresses of the plate. The problem is thus solved.

§10. CIRCULAR AND ANNULAR PLATES OF MINIMUM WEIGHT

We have solved the problems of bending of plates whose edges are reinforced by thin elastic ribs of different material. In this section we shall consider the influence of the stiffening rib on the weight of a symmetrically loaded circular or annular plate with reinforced edge, conducting the strength analysis for maximum normal stress.*

Using polar coordinates (r, θ) with origin at the center of the plate, we obtain from (2.51) and (2.52) along the joint Γ between plate and rib:

$$M_\theta = \frac{\nu\delta_1 \pm (1-\nu^2)}{\delta_1}M_r. \tag{3.218}$$

* The procedure can easily be adapted to other strength theories.

Here and henceforth the plus and minus signs correspond, respectively, to reinforcement of the inner or outer edge of the plate.

Hence, on Γ:

$$\left.\begin{aligned}\left|\frac{M_\theta}{M_r}\right| &> 1 \quad \text{for} \quad \delta_1 < (1 \pm \nu)\\ \left|\frac{M_\theta}{M_r}\right| &< 1 \quad \text{for} \quad \delta_1 > (1 \pm \nu)\end{aligned}\right\}. \tag{3.219}$$

The principal stresses σ_r, σ_θ, and σ_z at the surfaces of the plate are

$$\sigma_r = \pm \frac{6M_r}{h^2}, \quad \sigma_\theta = \pm \frac{6M_\theta}{h^2}, \quad \sigma_z = 0.$$

The plus and minus signs correspond, respectively, to the lower or upper surface of the plate.

Substituting these expressions in the formulas for the equivalent stresses (cf. /103/, Vol. 1, p. 319), we obtain

$$\sigma_{eq} = \frac{6f(M_r, M_\theta)}{h^2}, \tag{3.220}$$

in many theories of the limiting states of stress (maximum normal stresses, maximum linear strains, total potential energy, energy of shape alteration, etc.); here $f(M_r, M_\theta)$ is some function of the moments M_r, M_θ.

Let the stiffening rib be of rectangular section with sides b and h_1. The maximum normal stress in the rib is then

$$\sigma^*_{max} = \frac{|L_n| h_1}{2I} = \frac{E_1 h_1}{2D\delta_1} |M_r|_\Gamma, \tag{3.221}$$

where $I = \frac{bh_1^3}{12}$ and /135/ $L_n = R(M_r)_\Gamma$. Here L_n is the bending moment acting on the cross section of the rib.

The cross-sectional area of the rib is

$$F = bh_1 = \frac{12I}{h_1^2} = \frac{ER\delta_1}{E_1(1-\nu^2)} \cdot \frac{h^3}{h_1^2}. \tag{3.222}$$

It is evident from (3.101) and (3.125) that the bending moments acting on a plate are completely defined by the relative rigidity δ_1 of the rib. Thus, when δ_1 is fixed, the function $f(M_r, M_\theta)$ is constant [for r = const.].

The following conclusion can be drawn from (3.220) to (3.222). Among all reinforced plates of a given material and with a given relative rib rigidity δ_1, the structure (plate with rib) for which

$$\alpha_0 = k, \tag{3.223}$$

will be of minimum weight; here

$$\alpha_0 = \frac{\sigma^{max}_{eq}}{\sigma^*_{max}}, \quad k = \frac{[\sigma]}{[\sigma^*]},$$

where $[\sigma]$, $[\sigma^*]$ are the permissible normal stresses for the material of, respectively, plate and rib, $\sigma_{eq.}^{max}$ is the maximum value of the equivalent stress $\sigma_{eq.}$ given by (3.220) (/103/, Vol. I, 319).

To prove this assertion, we determine first α_0. We find from (3.220) and (3.221)

$$\alpha_0 = \frac{f_{max}(M_r, M_\theta)\cdot\delta_1}{|M_r|r}\cdot\frac{E}{E_1(1-\nu^2)}\cdot\frac{h}{h_1}. \tag{3.223a}$$

Consider the following two cases.

1) $\alpha_0 < k$. In this case, according to (3.223a), increasing α_0 to the value k is equivalent to increasing the ratio h/h_1. However, since δ_1 is given, $\frac{bh_1^3}{h^3}$ = const. and for constant cross-sectional area of the rib (i. e., for constant unit weight of the rib), $F = bh_1$ = const. Hence $\frac{h^3}{h_1^2}$ = const. Thus an increase in the ratio $\frac{h}{h_1}$ leads in this case to a reduction in the plate thickness h, i. e., to a lower weight of the structure.

2) $\alpha_0 > k$. At constant plate thickness h, we can reduce the ratio α_0 to the value k by increasing h_1, i. e., according to (3.222) by reducing the cross-sectional area of the rib. Thus, a decrease in α_0 leads in this case to a lower weight of the structure.

Thus, in both cases the structure is heavier than when $\alpha_0 = k$ ((3.223)). For a given value of δ_1, when $\alpha_0 = k$, the structure will thus be lightest.

We shall only consider the case when σ_{eq} is determined by the theory of maximum normal stresses. (Similar studies are possible for other theories of limiting states of stress.

We also assume that the maximum bending moments in the plate act along the contour reinforced by the elastic rib; this is the case in many problems discussed in the preceding sections of this chapter. We then obtain from (3.219) that on Γ.

$$\sigma_{eq.}^{max} = \sigma_{max} = \begin{cases} \frac{6|M_\theta|}{h^2} & \text{for} \quad \delta_1 \leqslant (1 \pm \nu), \\ \frac{6|M_r|}{h^2} & \text{for} \quad \delta_1 \geqslant (1 \pm \nu). \end{cases} \tag{3.224}$$

Substituting (3.224) and (3.221) in (3.223), and taking (3.218) into account, we obtain

$$\frac{h_1}{h} = \begin{cases} (1-\nu^2 \pm \nu\delta_1)/k\lambda(1-\nu^2) & \text{for} \quad \delta_1 \leqslant (1 \pm \nu), \\ \delta_1/k\lambda(1-\nu^2) & \text{for} \quad \delta_1 \geqslant (1 \pm \nu), \end{cases} \tag{3.225}$$

$$\frac{b}{R} = \begin{cases} k^3\lambda^2\delta_1(1-\nu^2)^2/(1-\nu^2 \pm \nu\delta_1)^3 & \text{for} \quad \delta_1 \leqslant (1 \pm \nu), \\ k^3\lambda^3(1-\nu^2)^2/\delta_1^2 & \text{for} \quad \delta_1 \geqslant (1 \pm \nu), \end{cases} \tag{3.226}$$

where $\lambda = \frac{E_1}{E}$.

Formulas (3.225) and (3.226) define the relative dimensions of a stiffening rib of relative flexural rigidity δ_1, which fulfills the minimum-weight condition (3.223).

Under the assumptions made here, these dimensions depend only indirectly on the actual axisymmetrical load acting on the plate: when the

maximum bending moments in the plate act along its reinforced edge Γ, the relative rib dimensions corresponding to the lightest structure are given by (3.225) and (3.226).

Figure 41 shows the magnitudes $\left(\frac{b}{R}\right):k^3\lambda^2$ and $\frac{h_1}{h}k\lambda$ as functions of $\alpha=\frac{\delta_1}{1+\nu+\delta_1}$ (curves 1 and 2, respectively), for a rib reinforcing the inner plate edge; Figure 42 represents the same curves for a rib reinforcing the outer plate edge. All curves have been plotted from (3.225) and (3.226) for $\nu=0.3$.

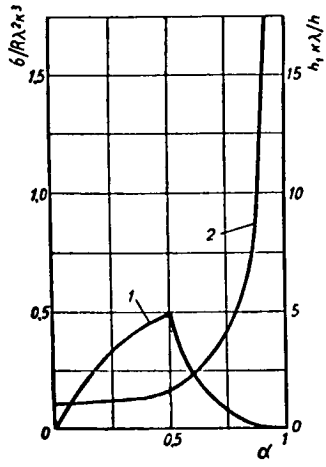


FIGURE 41.

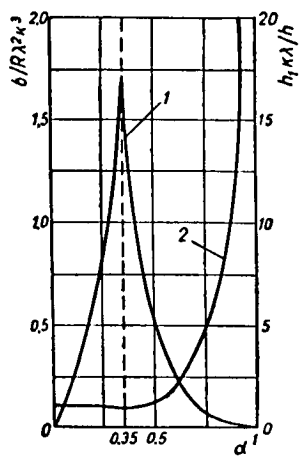


FIGURE 42.

We shall now determine the relative saving ΔQ in plate weight obtained through reinforcing the plate edge by a rib.

We introduce the following symbols: Q_0, h_0 are, respectively, weight and thickness of a nonreinforced plate under the given load; Q is the weight of the reinforced plate under the same load; γ_1 and γ_2 are the specific weights of plate and rib, respectively; R_0, R_1 are, respectively, the radii of the reinforced and of the nonreinforced plate edge; $R=R_0\pm\frac{b}{2}$ is the mean radius of the rib.

We have

$$\left.\begin{aligned} Q_0&=\pm\pi(R_1^2-R_0^2)h_0\gamma_1\\ Q&=\pm\pi(R_1^2-R_0^2)h\gamma_1+2\pi bR(h_1-h)\gamma_0 \end{aligned}\right\}. \tag{3.227}$$

Hence,

$$\Delta Q=\frac{Q_0-Q}{Q_0}=1-\frac{\beta}{c}, \tag{3.228}$$

where

$$\left.\begin{aligned}\beta &= 1 \pm 2\frac{b}{R}\left(\frac{h_1}{h}-1\right)\gamma\left[\eta^2-\left(\pm\frac{b}{2R}-1\right)^2\right]\\ \gamma &= \frac{\gamma_0}{\gamma_1}, \quad \eta = \frac{R_1}{R}, \quad c = \frac{h_0}{h}\end{aligned}\right\}. \tag{3.229}$$

The ratios b/R and h_1/h are given by (3.225) and (3.226).

Since

$$h = \sqrt{\frac{6M_{\max}}{[\sigma]}}, \qquad h_0 = \sqrt{\frac{6M^0_{\max}}{[\sigma]}},$$

where $M^0_{\max}$ is the maximum bending moment in the nonreinforced plate, we obtain

$$c = \sqrt{\frac{M^0_{\max}}{M_{\max}}}. \tag{3.230}$$

We thus obtain a plate of minimum weight by selecting δ_1 in such a way that ΔQ, defined by (3.228), is maximum.

Thus, for a plate with reinforced inner edge, supported on the outer or inner edge, we obtain from (3.101)

$$\left.\begin{aligned}M_r &= M_r^0 - \frac{(1-\nu^2)D\delta_1}{R\Delta_*}\left(\frac{R_1^2}{r^2}-1\right)\left(\frac{dw^0}{dr}\right)_\Gamma\\ M_\theta &= M_\theta^0 + \frac{(1-\nu^2)D\delta_1}{R\Delta_*}\left(\frac{R_1^2}{r^2}+1\right)\left(\frac{dw^0}{dr}\right)_\Gamma\end{aligned}\right\}. \tag{3.231}$$

The superscript "0" denotes the moments in the nonreinforced plate.

We assume that the maximum moments act along Γ. This is the case in all types of loading shown in Figure 43.

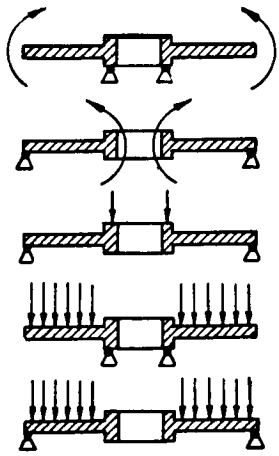


FIGURE 43.

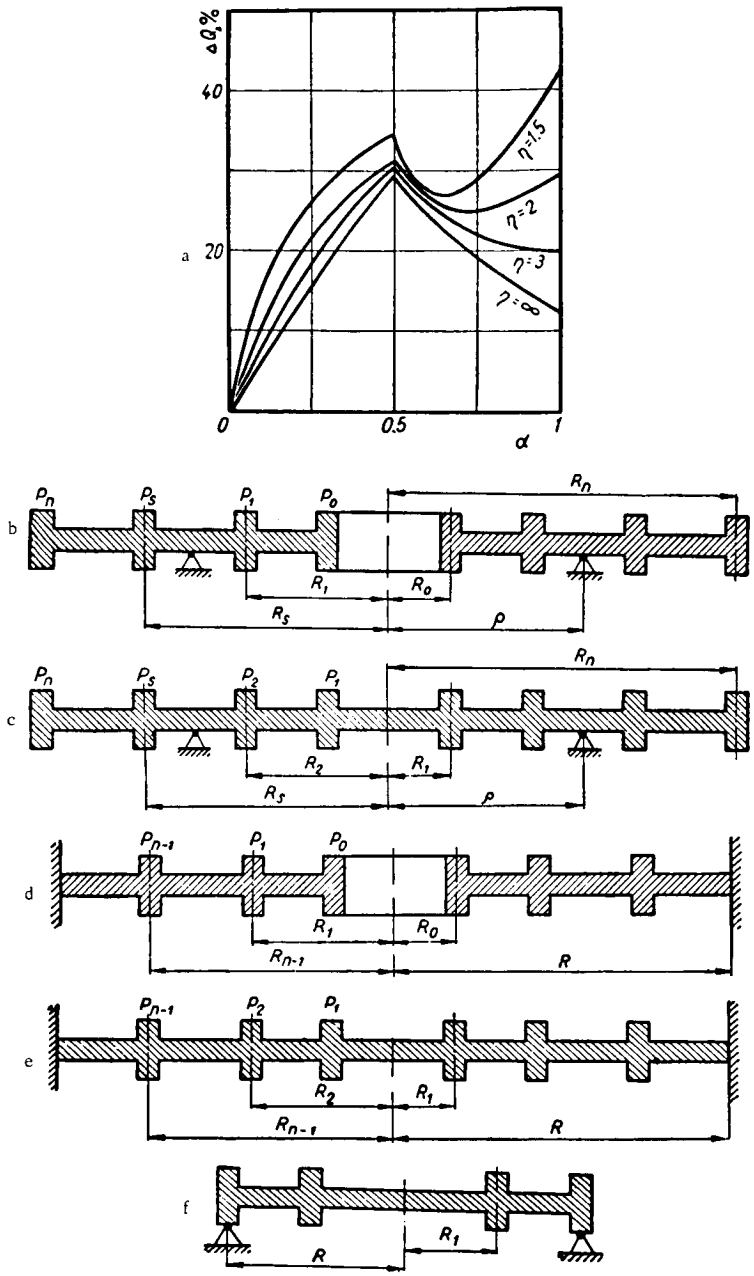

a
Δ0,%
40
20
0
0.5
1
α
η=1.5
η=2
η=3
η=∞
b
P_n
P_s
P_1
P_0
R_n
R_1
R_0
R_s
ρ
c
P_n
P_s
P_2
P_1
R_n
R_2
R_1
R_s
ρ
d
P_{n-1}
P_1
P_0
R_1
R_0
R_{n-1}
R
e
P_{n-1}
P_2
P_1
R_2
R_1
R_{n-1}
R
f
R_1
R

FIGURE 44.

Thus, according to (3.230) and (3.219),

$$c^2 = \frac{M^0_{\max}}{M_{\max}} = \begin{Bmatrix} (M^0_\theta/M_\theta)_\Gamma \text{ for } \delta_1 \leqslant (1+\nu) \\ (M^0_\theta/M_r)_\Gamma \text{ for } \delta_1 \geqslant (1+\nu) \end{Bmatrix}. \tag{3.232}$$

Since $M^0_r = 0$ on Γ, it follows that

$$\frac{d^2w^0}{dr^2} = -\frac{\nu}{R}\left(\frac{dw^0}{dr}\right)_\Gamma, \tag{3.233}$$

and therefore on Γ,

$$M^0_\theta = -\frac{D(1-\nu^2)}{R}\cdot\frac{dw^0}{dr}. \tag{3.234}$$

Substituting (3.234) and (3.231) in (3.232), we find

$$c^2 = \begin{cases} \Delta_*/(\eta^2-1)(1-\nu^2+\nu\delta_1) \\ \text{for } \delta_1 \leqslant (1+\nu), \\ \Delta_*/(\eta^2-1)\delta_1 \\ \text{for } \delta_1 \geqslant (1+\nu). \end{cases} \tag{3.235}$$

We obtain similarly from (3.125) for a plate rigidly clamped at its outer edge

$$c^2 = \begin{cases} (1-\nu^2)\Delta_{**}/(1-\nu^2+\nu\delta_1)[1+\nu+\eta^2(1-\nu)] \text{ for } \delta_1 \leqslant (1+\nu), \\ (1-\nu^2)\Delta_{**}/[1+\nu+\eta^2(1-\nu)]\delta_1 \text{ for } \delta_1 \geqslant (1+\nu). \end{cases} \tag{3.236}$$

Substituting c from (3.235) or (3.236) in (3.238), and taking (3.225) and (3.226) into account, we can plot ΔQ as function of δ_1. For the above example (with C given by (3.235)), this has been done in Figure 44,a where ΔQ is given in percentage as function of $\alpha = \frac{\delta_1}{(1+\nu+\delta_1)}$. The following values were used: $\nu = 0.3$, $\gamma = k = \lambda = 1$ (rib and plate of the same material).

Obviously, α should be selected so as to obtain ΔQ; the corresponding values of b/R and h_1/h (cf. Figures 41, 42) must lie witnin the limits for which the stiffening rib can be considered as thin.

If these limits are $\frac{b}{R} \leqslant 0.15$ and $\frac{h_1}{h} < 5$, we find from Figure 41: $0.64 \leqslant \alpha \leqslant 0.78$, i. e., $2.35 \leqslant \delta_1 \leqslant 4.55$. When the inner edge of the plate is reinforced by a rib of the same material, we find from Figure 44,a that the maximum saving possible for this range of variation of α is $\Delta Q_{\max} = 25.4\%$, the minimum saving being $\Delta Q_{\min} = 19.4\%$ for $\eta \geqslant 2$.

§ 11. AXISYMMETRICAL BENDING OF A CIRCULAR OR ANNULAR PLATE REINFORCED BY SEVERAL CONCENTRIC RIBS

An axisymmetrically loaded circular homogeneous annular plate of constant thickness is simply supported along $r = \varrho$ and reinforced along $r = R_j$ $(j = 0, 1, 2, \ldots, n)$, by several concentric thin elastic ribs (Figure 44,b).

For $R_0 \leqslant r \leqslant R_1$ the deflections are assumed to be

$$w_1(r) = C_3 + C_4\left(\frac{r^2}{\varrho^2} - 1\right) + C_1 \ln\frac{r}{\varrho} + w^0, \tag{3.237}$$

where w^0 is the basic deflection of the nonreinforced plate under the same conditions of loading and support. The function $w^0(r)$ is assumed to be known. For $R_1 \leqslant r \leqslant R_2$ the deflections are represented in the form

$$w_2(r) = w_1(r) + k_1\left(\frac{r^2}{R_1^2} - 1 - 2\ln\frac{r}{R_1}\right), \tag{3.238}$$

where k_1 is a constant to be determined from the condition that the two plate domains are joined along $r = R_1$. In the case considered, the last three conditions (2.368) are satisfied automatically, while the first becomes

$$\delta_1 \frac{dw_1}{dr} + R_1 \frac{d^2}{dr^2}(w_1 - w_2) = 0 \quad \text{for } r = R_1, \tag{3.239}$$

where $\delta_1 = \frac{A_1}{R_1 D}$ is the relative flexural rigidity of rib P_1. Substituting (3.237) and (3.238) in (3.239), we obtain

$$k_1 = \frac{1}{4} R_1 \delta_1 \left(\frac{dw_1}{dr}\right)_{r=R_1}. \tag{3.240}$$

We find by mathematical induction that for $R_{j-1} \leqslant r \leqslant R_j$ $(j = 1, 2, 3, \ldots, n)$ the deflections can be written in the form

$$w_j(r) = C_3 + C_4\left(\frac{r^2}{\varrho^2} - 1\right) + C_1 \ln\frac{r}{\varrho} + w^0 + \frac{1}{4}\sum_{k=0}^{j-1}\left(\frac{r^2}{R_k^2} - 1 - 2\ln\frac{r}{R_k}\right)\theta_k \delta_k, \tag{3.241}$$

where

$$\theta_0 = 0, \quad \theta_k = R_k\left(\frac{dw_k}{dr}\right)_{r=R_k}, \quad k > 0, \quad \delta_k = \frac{A_k}{R_k D}. \tag{3.242}$$

Substitution of (3.241) in (3.242) yields the following recurrent formula for the constants θ_k $(k > 0)$:

$$\theta_k = R_k \gamma_k + C_1 + 2C_4 \frac{R_k^2}{\varrho^2} + \sum_{i=0}^{k-1} \frac{1}{2}\theta_i \delta_i \left(\frac{R_k^2}{R_i^2} - 1\right), \tag{3.243}$$

where

$$\gamma_k = \left(\frac{dw^0}{dr}\right)_{r=R_k}. \tag{3.244}$$

Setting

$$m_{ij} = \frac{1}{2}\left(\frac{R_j^2}{R_i^2} - 1\right)\delta_{ij}, \tag{3.245}$$

we can rewrite (3.243) in the form

$$\theta_k = \sum_{i=1}^{k}\sum_{j=i+1}^{k} m_{ij}\left(R_i\gamma_i + C_1 + 2C_4\frac{R_i^2}{\varrho^2}\right), \tag{3.246}$$

with

$$m_{ij} = 1 \text{ for } i = k. \tag{3.247}$$

Rearranging terms in (3.246), we write

$$\frac{1}{4}\theta_k\delta_k = \alpha_{0k} + \alpha_{1k}C_1 + \alpha_{4k}C_4, \tag{3.248}$$

where

$$\left.\begin{aligned} \alpha_{1k} &= \frac{\delta_k}{4}\sum_{i=1}^{k}\sum_{j=i+1}^{k} m_{ij}, & \alpha_{4k} &= \frac{\delta_k}{2}\sum_{i=1}^{k}\sum_{j=i+1}^{k} m_{ij}\frac{R_j^2}{\varrho^2} \\ \alpha_{0k} &= \frac{\delta_k}{4}\sum_{i=1}^{k}\sum_{j=i+1}^{k} m_{ij}R_i\gamma_i, & \alpha_{00} &= \alpha_{10} = \alpha_{40} = 0 \end{aligned}\right\}. \tag{3.249}$$

Using (3.248), we write (3.241) in the form

$$w_j(r) = C_3 + C_4\left(\frac{r^2}{\varrho^2} - 1\right) + C_1\ln\frac{r}{\varrho} + w^0 + \sum_{k=0}^{j-1}(\alpha_{0k} + \alpha_{1k}C_1 + \alpha_{4k}C_4)\Phi_k(r), \tag{3.250}$$

where

$$\Phi_k(r) = \frac{r^2}{R_k^2} - 1 - 2\ln\frac{r}{R_k}.$$

The constants C_3, C_1 and C_4 are determined from the boundary conditions for $r = R_0$ and $r = R = R_n$, and from the support condition

$$w_s = 0 \text{ for } r = \varrho, \tag{3.251}$$

where s is the ordinal number of the region in which the support is located. Substituting (3.250) in (3.251), we obtain

$$C_3 = -\sum_{k=0}^{s-1} (\alpha_{0k} + \alpha_{1k} C_1 + \alpha_{4k} C_4)\, \Phi_k(\varrho). \tag{3.252}$$

Two of the four boundary conditions (2.366) and (2.367) are satisfied automatically in the case considered, and only the following remain:

$$\left.\begin{aligned} (\delta_n + \nu)\frac{dw}{dr} + R\frac{d^2w}{dr^2} = 0 \text{ for } r = R \\ (\delta_0 - \nu)\frac{dw}{dr} - R_0\frac{d^2w}{dr^2} = 0 \text{ for } r = R_0 \end{aligned}\right\}. \tag{3.252a}$$

This yields the equations

$$\left.\begin{aligned} a_1 C_4 + b_1 C_1 + d_1 = 0 \\ a_2 C_4 + b_2 C_1 + d_2 = 0 \end{aligned}\right\}, \tag{3.253}$$

where

$$\left.\begin{aligned} a_1 &= 2\frac{R^2}{\varrho^2}(1 + \nu + \delta_n) + \sum_{k=0}^{n-1}\sum_{i=1}^{k}\sum_{j=i+1}^{k} m_{ij}\delta_k \Delta_{nk}\frac{R_i^2}{\varrho^2} \\ b_1 &= (\delta_n + \nu - 1) + \frac{1}{2}\sum_{k=0}^{n-1}\sum_{i=1}^{k}\sum_{j=i+1}^{k} m_{ij}\delta_k \Delta_{nk} \\ d_1 &= R\gamma_n\delta_n + \frac{1}{2}\sum_{k=0}^{n-1}\sum_{i=1}^{k}\sum_{j=i+1}^{k} m_{ij}\delta_k\Delta_{nk} R_i\gamma_i \\ a_2 &= 2(\delta_0 - \nu - 1)\frac{R_0^2}{\varrho^2}, \quad b_2 = \delta_0 - \nu + 1, \quad d_2 = R_0\gamma_0 \\ \Delta_{nk} &= (\delta_n + 1 + \nu)\frac{R^2}{R_k^2} - (\delta_n - 1 + \nu) \end{aligned}\right\} \tag{3.254}$$

with $\Delta_{nk} = 0$ for $k = 0$.

The solution of system (3.253) is

$$C_1 = \frac{d_1 a_2 - d_2 a_1}{a_1 b_2 - b_1 a_2}, \qquad C_4 = \frac{d_2 b_1 - d_1 b_2}{a_1 b_2 - b_1 a_2}. \tag{3.255}$$

The problem is thus solved. Knowing the coefficients (3.255) and (3.252), we can determine the deflections and, by (1.19), the bending moments at any point of the plate. We present some examples:

1) A solid circular plate is reinforced by thin elastic ribs (Figure 44,c). This problem is solved easily, setting $R_0 = 0$ and $C_1 = 0$ in the above formulas:

$$w_j(r) = w^0 + C_3 + C_4\left(\frac{r^2}{\varrho^2} - 1\right) + \sum_{k=0}^{j-1} (\alpha_{0k} + \alpha_{4k} C_4)\, \Phi_k(r), \tag{3.256}$$

where

$$C_4 = -\frac{d_1}{a_1},\quad C_3 = -\sum_{k=0}^{s-1}(\alpha_{0k} + \alpha_{4k}C_4)\,\Phi_k(\varrho). \tag{3.257}$$

2) A circular annular plate with concentric ribs is rigidly clamped along $r = R_n = R$ (Figure 44,d). Letting $\delta_n \to \infty$ and writing $\varrho = R$ in (3.255), we obtain

$$C_1 = \frac{d_1'a_2' - d_2'a_1'}{a_1'b_2' - b_1'a_2'},\quad C_4 = \frac{d_2'b_1' - d_1'b_2'}{a_1'b_2' - b_1'a_2'}, \tag{3.258}$$

where

$$\left.\begin{aligned} a_1' &= 2 + \sum_{k=0}^{n-1}\sum_{i=1}^{k}\sum_{j=i+1}^{k} m_{ij}\delta_k\left(\frac{R^2}{R_k^2} - 1\right)\frac{R_i^2}{R^2} \\ b_1' &= 1 + \frac{1}{2}\sum_{k=0}^{n-1}\sum_{i=1}^{k}\sum_{j=i+1}^{k} m_{ij}\delta_k\left(\frac{R^2}{R_k^2} - 1\right) \\ d_1' &= R\gamma_n + \frac{1}{2}\sum_{k=0}^{n-1}\sum_{i=1}^{k}\sum_{j=i+1}^{k} m_{ij}\delta_k R_i\gamma_i\left(\frac{R^2}{R_k^2} - 1\right) \\ b_2' &= \delta_0 - \nu + 1, \qquad d_2' = R_0\gamma_0, \quad a_2' = 2(\delta_0 - \nu - 1)\frac{R_0^2}{R^2} \end{aligned}\right\}. \tag{3.259}$$

From (3.252) we find

$$C_3 = -\sum_{k=0}^{n-1}(\alpha_{0k} + \alpha_{1k}C_1 + \alpha_{4k}C_4)\,\Phi_k(R). \tag{3.260}$$

The deflections $w_j(r)$ are given, as before, by (3.250), where w^0 is the deflection of the nonreinforced plate hinged along $r = R$, and must be replaced by R. If w^0 is the deflection of a nonreinforced plate clamped at the edge, we must set $\gamma_n = 0$ in (3.259).

3) A solid circular plate reinforced by ribs is rigidly clamped along $r = R_n = R$ (Figure 44,e). In this case the deflections are given by (3.256) for $\varrho = R$, while the coefficients C_3 and C_4 are obtained from (3.257) by letting $\delta_n \to \infty$ and writing $\varrho = R$:

$$C_3 = -\sum_{k=0}^{n-1}(\alpha_{0k} + \alpha_{4k}C_4)\,\Phi_k(R), \qquad C_4 = -\frac{d_1'}{a_1'}. \tag{3.261}$$

4) A circular plate reinforced by two stiffening ribs is hinged along $r = R$ (Figure 44,f). Substituting in (3.256) and (3.257) $\varrho = R$ and $s = n = 2$, we find for $0 < r < R_1$

$$w_1(r) = w^0 + C_3 + C_4\left(\frac{r^2}{R^2} - 1\right),$$

for $R_1 < r < R$

$$w_2(r) = w^0 + C_3 + C_4\left(\frac{r^2}{R^2} - 1\right) + \frac{\delta_1}{4}\left(R_1\gamma_1 + 2\frac{R_1^2}{R^2}C_4\right)\left(\frac{r^2}{R_1^2} - 1 - 2\ln\frac{r}{R_1}\right), \quad (3.262)$$

where

$$\left.\begin{aligned} C_3 &= -\frac{\delta_1}{4}(R_1\gamma_1 + 2\eta_1^{-2}C_4)(\eta_1^2 - 1 - 2\ln\eta_1) \\ C_4 &= \frac{\delta_1 R_1\gamma_1[(\delta_2 + \nu - 1) - \eta_1^2(\delta_2 + \nu + 1)] - 2R\delta_2\gamma_2}{4(\delta_2 + \nu + 1) + 2\delta_1\eta_1^{-2}[(\delta_2 + \nu + 1)\eta_1^2 - (\delta_2 + \nu - 1)]} \\ \eta_1 &= \frac{R}{R_1} \end{aligned}\right\}. \quad (3.263)$$

Solutions for several cases of axisymmetrical loading of a circular plate with two stiffening ribs were obtained in /31/ by a different procedure.

Formulas for the deflections can be easily derived for many other particular cases.

Chapter IV

PLATES REINFORCED BY VERY THIN RIBS

§ 1. STATEMENT OF THE PROBLEM

Problems on the reinforcement of plates along the edges of holes have been solved (cf. /8, 34, 114-118, 122-125, 187/) under the assumption that the rib, of a different material, which reinforces the edge of the hole is so wide that its stresses and strains can be analyzed according to the two-dimensional theory of elasticity or by the equations of bending of thin plates, depending on the type of load applied to the plate.

This assumption is invalid if the stiffening rib is very thin or has an irregular profile. In this case the stiffening rib should be considered as a thin curved bar whose strains are described by the theory of small deformations of thin curved bars.

Sheremet'ev (/180—184/) derived the boundary conditions for a plate having a rib of uniform cross section welded around the curvilinear edge of a hole. He considered the stiffening rib to be a curved bar, possessing either tensile and flexural rigidity in the case of a generalized plane state of stress, or flexural and torsional rigidity in the case of bending like a thin plate.

Solutions were obtained quite easily in problems on the elastic equilibrium of plates having circular holes with edges reinforced by thin ribs. The analysis of plates having holes with curvilinear (noncircular) reinforced edges presents, however, considerable difficulties (/71-74, 186-188/). It is therefore advisable to simplify the formulation of the problem.

Following /153/ and /159/ we derive below approximate boundary conditions for the joint between a plate and a thin curved elastic bar of variable cross section, made of a different material. It is assumed that the bar axis Γ lies in the middle (xy)-plane of the plate, and that contact between bar and plate takes place along the cylindrical surface S which passes through Γ and is normal to the (xy)-plane. When Γ is described in the positive sense, the domain of the plate lies to the left.

Instead of requiring the exact fulfilment of all conditions of joining, we require that the following conditions only be satisfied on S:

I) The forces and moments, per unit length of Γ, acting between plate and bar satisfy Newton's Third Law.

II) The strains ε_τ and ε_0 of fibers, respectively of the plate and of the bar, equidistant from the xy-plane are equal.

In addition the stiffening rib is considered to be so thin that the following assumptions can be made:

1) In the case of bending like a thin plate we can neglect torques acting on the rib cross sections, i. e., we assume that the rib possesses only flexural rigidity, and that one of the principal axes of inertia of the rib cross section lies in the xy-plane; the Z-axis is directed downward (Figure 45).

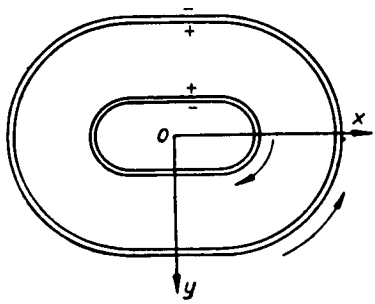


FIGURE 45.

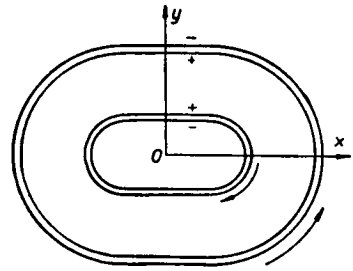


FIGURE 46.

2) In the case of a generalized plane state of stress we can neglect bending moments acting on the rib cross sections, i. e., consider the bar to be an elastic line possessing only tensile or compressive rigidity* (Figure 46). We shall derive below the approximate boundary conditions corresponding to these assumptions.

§ 2. BENDING OF A THIN PLATE. DERIVATION OF THE BOUNDARY CONDITION FOR THE REINFORCED EDGE

Conditions I and II of the preceding section are written in the following form for a thin plate:

$$\int_0^s \left[M_n - i \int_0^{s_1} \left(N_n + \frac{dH_{n\tau}}{ds_2} \right) ds_2 \right] \bar{t}\, ds_1 = \int_0^s \left[m_1 - i \int_0^{s_1} p_1 ds_2 \right] \bar{t}\, ds_1 \text{ on } \Gamma \tag{4.1}$$

$$\varepsilon_\tau = \varepsilon_0 \text{ for all } Z \text{ on } S, \tag{4.2}$$

where M_n, $H_{n\tau}$, N_n are, respectively, the bending moment, twisting moment, and shearing force acting in the plate along Γ per unit of its length; m_1 and p_1 are, respectively, the bending moment and the shearing force with which the plate acts on unit length of the rib; $t = x + iy$ is the affix of a point on . s, s_1, s_2 are arc lengths measured from the origin 0; $\dot{t} = \frac{dt}{ds}$; n is the outer normal to Γ.

* It was shown by Sheremet'ev /184/ that the flexural rigidity has in this case only a small influence on the stresses in the plate.

Proceeding from the equilibrium conditions for part Os of the rib, assuming that no torques act on the rib cross sections, we obtain from (2.22) for $L_\tau = 0$:

$$iL_n\dot{\bar{t}} = \int_0^s [m(s_1) - i\int_0^{s_1} p(s_2)\,ds_2]\dot{\bar{t}}\,ds_1 + iC\bar{t} - C', \tag{4.3}$$

where

$$m(s) = -m^* + m_1, \qquad p(s) = -p^* + p_1, \tag{4.4}$$

m^* and p^* are given external bending moments and shearing forces acting on the ring; L_n is the bending moment in section s of the rib; C' is a complex constant; C is a real constant.

In addition, we can express the strains ε_0 and ε_τ by the known formulas of strength of materials (/2/, p. 265) and of the theory of elasticity (/61/, p. 17):

$$\varepsilon_0 = ZL_n/A, \tag{4.5}$$

$$\varepsilon_\tau = \frac{\partial u}{\partial x}\dot{x}^2 + \frac{\partial v}{\partial y}\dot{y}^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\dot{x}\dot{y}, \tag{4.6}$$

where u, v are the displacement components in the x- and y-directions; $A = E_1 I$ is the flexural rigidity of the rib (variable in the general case); E_1 is Young's modulus for the rib, $\dot{x} = \frac{dx}{ds}$, $\dot{y} = \frac{dy}{ds}$.

Substituting (1.1), we rewrite (4.6) in the form

$$\varepsilon_\tau = -Z\left[\frac{\partial^2 w}{\partial x^2}\dot{x}^2 + 2\frac{\partial^2 w}{\partial x \partial y}\dot{x}\dot{y} + \frac{\partial^2 w}{\partial y^2}\dot{y}^2\right] = -\operatorname{Re}\left[i\frac{d}{ds}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right)\right]Z. \tag{4.7}$$

Using (4.3), (4.5), and (4.7), we eliminate the unknown right-hand sides of (4.1) and (4.2) and obtain a single approximate boundary condition in complex form for a plate with reinforced edge:

$$-\int_0^s \left[M_n - i\int_0^{s_1}\left(N_n + \frac{d}{ds_2}H_{n\tau}\right)ds_2\right]\dot{\bar{t}}\,ds_1 -$$

$$-\dot{\bar{t}}A\operatorname{Re}\left[i\frac{d}{ds}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right)\right] = l^0(s) + iC\bar{t} - C', \tag{4.8}$$

where

$$l^0(s) = -\int_0^s (m^* - i\int_0^{s_1} p^* ds_2)\,\dot{\bar{t}}\,ds_1. \tag{4.9}$$

is known. We write (cf. (1.23)):

$$w = 2\operatorname{Re}[\bar{z}\varphi(z) + \chi(z)] + w^0(x, y), \tag{4.10}$$

where $\varphi(z)$ and $\chi(z)$ are analytic functions of the complex variable $z = x + iy$; $w^0(x, y)$ is any particular solution of the equation (cf. (1.13)) $\Delta\Delta w = q/D$.

The terms on the left-hand side of (4.8) can be expressed through the functions $\varphi(z)$ and $\psi(z) = \int \chi(z)\, dz$ in accordance with (1.65) and (1.76). We then obtain instead of (4.8):

$$-\frac{iA\bar{t}}{D(1-\nu)}\operatorname{Re}\left\{t\frac{d}{ds}\left[\overline{\varphi(t)} + \bar{t}\varphi'(t) + \psi(t)\right]\right\} + \\ + \varkappa\overline{\varphi(t)} - \bar{t}\varphi'(t) - \psi(t) = f_1(t) + iC_1'\bar{t} - C_2', \tag{4.11}$$

where $\varkappa = (3+\nu)/(1-\nu)$; ν is Poisson's ratio; C_1', C_2' are, respectively, real and complex integration constants;

$$\left.\begin{aligned} f_1(t) &= -\frac{1}{2D(1-\nu)}[I_0(s) - I^0(s)] + \\ &+ \frac{\dot{i}\dot{t}A}{2D(1-\nu)}\operatorname{Re}\left\{t\frac{d}{ds}\left(\frac{\partial w^0}{\partial x} - i\frac{\partial w^0}{\partial y}\right)\right\} \\ I_0(s) &= -\int_0^s (m_0 - i\int_0^{s_1} p_0 ds_2)\dot{\bar{t}}\, ds_1 \\ m_0 &= -D\left\{\nu\Delta w^0 + (1-\nu)\left[\frac{\partial^2 w^0}{\partial x^2}\cos^2\theta + \frac{\partial^2 w^0}{\partial y^2}\sin^2\theta + \right.\right. \\ &\left.\left. + 2\frac{\partial^2 w^2}{\partial x \partial y}\cos\theta\sin\theta\right]\right\} \\ p_0 &= D\left\{\frac{d\Delta w^0}{dn} + \left(1-\nu\right)\frac{d}{ds}\left[\left(\frac{\partial^2 w^0}{\partial y^2} - \frac{\partial^2 w^0}{\partial x^2}\right)\cos\theta\sin\theta + \right.\right. \\ &\left.\left. + \frac{\partial^2 \chi^0}{\partial x \partial y}\cos 2\theta\right]\right\} \end{aligned}\right\}, \tag{4.12}$$

θ is the angle between the normal n to Γ and the x-axis.

The degree of determinacy of the functions $\varphi(z)$ and $\psi(z)$ is obviously the same as in the first basic problem of bending of a thin plate.

In the case of axisymmetrical bending the boundary condition (4.11) is exact, since the twisting moments in the rib sections vanish identically.

The normal stress in a rib section is $\sigma_\tau^0 = E_1\varepsilon_0$.

§ 3. GENERALIZED STATE OF PLANE STRESS. DERIVATION OF THE BOUNDARY CONDITION FOR THE REINFORCED EDGE

Conditions I and II of § 1 become in this case:

$$\int_0^s (X_n + iY_n)\, ds = -\frac{1}{h}\int_0^s (X_n^0 + iY_n^0)\, ds \text{ on } \Gamma, \tag{4.13}$$

$$\varepsilon_\tau = \varepsilon_0 \text{ on } \Gamma, \tag{4.14}$$

where X_n, Y_n are the stresses with which the rib acts on the plate along Γ; X_n^0, Y_n^0 are the projections of the forces per unit length of rib axis with which the plate acts on the rib; h is the thickness of the plate.

Assuming that no bending moments act on the cross sections of the rib, the equilibrium condition for part Os of the rib becomes:

$$-\int_0^s (X_n^0 + iY_n^0)\, ds = \int_0^s (P_x + iP_y)\, ds + N\dot{t} - C_3, \tag{4.15}$$

where N is the normal tensile force acting in section s of the rib; C_3 is a complex constant; P_x, P_y, are the projections of the given external load per unit length of the rib axis.

We have, in accordance with (4.6),

$$\varepsilon_\tau = \operatorname{Re}\left[\bar{t}\,\frac{d}{ds}(u + iv)\right]. \tag{4.16}$$

Using (4.15) and (4.16), and noting that

$$\varepsilon_0 = N/E_1F, \tag{4.17}$$

(E_1 is Young's modulus for the rib; F is the cross-sectional area of the rib, variable in the general case), we eliminate the unknown right-hand sides of (4.13) and (4.14), and obtain the following single approximate boundary condition for the plate:

$$\int_0^s (X_n + iY_n)\, ds - \frac{\dot{t}E_1F}{h} \operatorname{Re}\left[\bar{t}\,\frac{d}{ds}(u + iv)\right] = \frac{1}{h}\int_0^s (P_x + iP_y)\, ds - C_3' \text{ on } \Gamma. \tag{4.18}$$

We can express the left-hand side of this equation through the functions $\varphi(z)$ and $\psi(z)$ ($z = x + iy$), holomorphic in the domain of the plate (/82/, pp. 113-114):

$$\left.\begin{aligned} \varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} &= -i\int_0^s (X_n + iY_n)\, ds + \text{const} \\ \varkappa_1\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} &= 2\mu(u + iv) \end{aligned}\right\}, \tag{4.19}$$

where t is the affix of a point on Γ; $\mu = E/2(1 + \nu)$ is the shear modulus; E is Young's modulus for the plate; $\varkappa_1 = (3 - \nu)/(1 + \nu)$; ν is Poisson's ratio.

Substituting (4.19) in (4.18), we arrive at the following boundary-value problem for the functions $\varphi(z)$ and $\psi(z)$, after introducing complex conjugate magnitudes:

$$\begin{aligned} &\frac{iE_1F}{2\mu h}\,\dot{t}\operatorname{Re}\left\{\dot{t}\,\frac{d}{ds}\left[\varkappa_1\overline{\varphi(t)} - \bar{t}\varphi'(t) - \psi(t)\right]\right\} + \\ &+ \overline{\varphi(t)} + \bar{t}\varphi'(t) + \psi(t) = f_1(t) + \text{const on } \Gamma, \end{aligned} \tag{4.20}$$

where

$$f_1(t) = -\frac{i}{h}\int_0^s (P_x - iP_y)\,ds. \tag{4.21}$$

The degree of determinacy of the functions $\varphi(z)$ and $\psi(z)$ is the same as in the first basic problem of the two-dimensional theory of elasticity.

The normal stresses $\sigma_\tau^0 = E_1\varepsilon_0$ in the rib are by (4.14), (4.16), and (4.19):

$$\sigma_\tau^0 = \frac{E_1}{E}(1+\nu)\,\mathrm{Re}\left\{i\frac{d}{ds}[\varkappa_1\overline{\varphi(t)} - \bar{t}\varphi'(t) - \psi(t)]\right\}. \tag{4.22}$$

The problem considered here was solved in /212/ by reducing it to a more involved boundary condition. An advantage of (4.20) is its similarity to the approximate boundary condition (4.11) for a thin reinforced plate. The methods of solving these problems are therefore also similar.

§ 4. UNIFIED CONTACT PROBLEM WITH APPROXIMATE BOUNDARY CONDITIONS

By adding the boundary conditions (4.11) and (4.20), corresponding, respectively, to the bending of a thin plate and to the generalized state of plane stress of a thin plate, we obtain the following boundary-value problem:

$$iK\dot{t}\,\mathrm{Re}\left\{i\frac{d}{ds}[a_2\overline{\varphi(t)} - \bar{t}\varphi'(t) - \psi(t)]\right\} +$$
$$+ a_1\overline{\varphi(t)} + \bar{t}\varphi'(t) + \psi(t) = a_3 f_1(t) - iC_1'\bar{t} + C_2' \text{ on } \Gamma. \tag{4.23}$$

In the case of bending of a thin plate,

$$a_1 = -\frac{3+\nu}{1-\nu}, \qquad a_2 = a_3 = -1, \qquad K = -\frac{A}{D(1-\nu)} < 0, \tag{4.24}$$

while the function $f_1(t)$ is defined by (4.12).

For a generalized state of plane stress,

$$a_1 = 1, \qquad a_2 = \frac{3-\nu}{1+\nu}; \qquad a_3 = 1, \qquad K = \frac{E_1F}{2\mu h} > 0, \qquad C_1' = 0, \tag{4.25}$$

the function $f_1(t)$ being defined by (4.21).

In (4.23), Γ denotes the contour along which the thin plate is joined to a thin rib whose properties are defined in § 1; $K = K(t)$ is a real function of the point t on Γ, which depends on the relative rigidity of the stiffening rib. The function K will sometimes be simply called rigidity.

Rewriting (4.23) in complex conjugate magnitudes, we obtain

$$-iK\dot{t}\,\mathrm{Re}\left\{\dot{\bar{t}}\frac{d}{ds}[a_2\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}]\right\} + a_1\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f_0(t) \text{ on } \Gamma, \tag{4.26}$$

where

$$f_0(t) = a_3\overline{f_1(t)} + iC_1't + \overline{C_2'}. \tag{4.27}$$

Setting

$$U(t) = t\overline{\varphi'(t)} + \overline{\psi(t)}, \tag{4.28}$$

we rewrite (4.26) in the following form:

$$K\,\mathrm{Re}\left\{\dot{\bar{t}}\,\frac{d}{ds}[a_1\varphi(t) - U(t)]\right\} + i\dot{\bar{t}}\,[a_1\varphi(t) + U(t)] = i\dot{\bar{t}}f_0(t). \tag{4.29}$$

Equating the imaginary parts on either side of (4.29), we obtain

$$\mathrm{Im}\,\{i\dot{\bar{t}}[a_1\varphi(t) + U(t)]\} = \mathrm{Im}\,\{i\dot{\bar{t}}f_0(t)\}, \tag{4.30}$$

or

$$i\dot{\bar{t}}[a_1\varphi(t) + \overline{U(t)}] + i\dot{t}[a_1\overline{\varphi(t)} + U(t)] = 2i\,\mathrm{Im}\,[f_0(t)\,i\dot{\bar{t}}], \tag{4.31}$$

whence

$$\overline{U(t)} = -a_1\varphi(t)\dot{\bar{t}}^2 - a_1\overline{\varphi(t)} - \dot{\bar{t}}^2U(t) + 2\dot{\bar{t}}\,\mathrm{Im}\,[f_0(t)\,i\dot{\bar{t}}].$$

Differentiating, we obtain

$$\frac{d}{ds}\overline{U(t)} = -a_1\dot{\bar{t}}^2\frac{d}{ds}\varphi(t) - 2a_1\dot{\bar{t}}\,\ddot{\bar{t}}\,\varphi(t) - a_1\frac{d}{ds}\overline{\varphi(t)} -$$

$$-\dot{\bar{t}}^2\frac{d}{ds}U(t) - 2U(t)\,\dot{\bar{t}}\,\ddot{\bar{t}} + 2\frac{d}{ds}\{\dot{\bar{t}}\,\mathrm{Im}\,[i\dot{\bar{t}}f_0(t)]\}. \tag{4.32}$$

We rewrite (4.29) as follows:

$$-iK\dot{t}\left\{a_2\dot{\bar{t}}\frac{d}{ds}\varphi(t) - \dot{\bar{t}}\frac{d}{ds}U(t) + a_2\dot{t}\frac{d}{ds}\overline{\varphi(t)} - \dot{t}\frac{d}{ds}\overline{U(t)}\right\} +$$

$$+ 2[a_1\varphi(t) + U(t)] = 2f_0(t). \tag{4.33}$$

Substitution of (4.32) in (4.33) yields

$$a_1a_2(t)\,\varphi(t) + a_2(t)\,U(t) - a_3(t)\left[\frac{d}{ds}\varphi(t) + \dot{t}^2\frac{d}{ds}\overline{\varphi(t)}\right] = f_*(t), \tag{4.34}$$

where

$$a_2(t) = 2(1 - iK\dot{t}\,\ddot{\bar{t}}),$$

$$a_3(t) = iK(a_1 + a_2),$$

$$f_*(t) = 2f_0(t) - 2iK\dot{t}^2\frac{d}{ds}[\dot{\bar{t}}\,\mathrm{Im}\,\{f_0(t)\,i\dot{\bar{t}}\}] = 2\overline{f_1(t)}\,a_3 - \tag{4.35}$$

$$-2iK\dot{t}^2a_3\frac{d}{ds}\{\dot{\bar{t}}\,\mathrm{Im}\,[i\dot{\bar{t}}\,\overline{f_1(t)}]\} + a_2(t)\,(C_2' + iC_1't).$$

Since

$$i\dot{t}\ddot{\bar{t}} = (\dot{x} + i\dot{y})(\ddot{x} - i\ddot{y}) = \mp\frac{i}{\rho},$$

where $\rho = \rho(t)$ is the variable radius of curvature of Γ, we have

$$a_2(t) = 2\left(1 \mp \frac{K}{\rho}\right), \tag{4.36}$$

where the sign before the last term is determined as follows:

TABLE 15

Type of problem	Region occupied by plate	Sign
Bending of plate	Exterior of Γ	Upper
	Interior of Γ	Lower
Two-dimensional	Exterior of Γ	Lower
	Interior of Γ	Upper

Substitution of (4.28) in (4.34) yields

$$a_3(t)[a_1\psi(t) + i\overline{\varphi'(t)} + \overline{\psi(t)}] - a_3(t)\dot{t}[\varphi'(t) + \overline{\varphi'(t)}] = f_*(t) \text{ on } \Gamma. \tag{4.37}$$

This is the transformed approximate boundary condition of the unified contact problem. When the plate edge is not reinforced, i.e., when $K = 0$, we have $a_3(t) = 0$, $a_2(t) = 2$, $f_*(t) = 2f_0(t)$, and we obtain from (4.37) the boundary condition for the first basic problem of the two-dimensional theory of elasticity.

We write the unknown functions $\varphi(z)$ and $\psi(z)$ in the form

$$\varphi(z) = \varphi^0(z) + \varphi_*(z), \qquad \psi(z) = \psi^0(z) + \psi_*(z), \tag{4.38}$$

where $\varphi^0(z)$ and $\psi^0(z)$ represent the known solution of the problem for the non-reinforced plate edge ($K = 0$); $\varphi_*(z)$ and $\psi_*(z)$ are unknown functions, characterizing the influence of the stiffening rib, which are holomorphic everywhere in the domain of the plate.

Substituting (4.38) in (4.37), we obtain the boundary condition on Γ:

$$a_2(t)[a_1\varphi_*(t) + i\overline{\varphi_*'(t)} + \overline{\psi_*(t)}] - a_3(t)\dot{t}[\varphi_*'(t) + \overline{\varphi_*'(t)}] = f_{**}(t), \tag{4.39}$$

where

$$f_{**}(t) = f_*(t) + a_3(t)\,\dot{t}[\varphi^{0\prime}(t) + \overline{\varphi^{0\prime}(t)}] - f_0(t)\,a_2(t) =$$

$$= iK\dot{t}\{2\ddot{\bar{t}}\,\overline{f_1(t)}a_3 - 2a_3\dot{t}\frac{d}{ds}(\dot{\bar{t}}\,\mathrm{Im}[i\dot{t}\,\overline{f_1(t)}]) + (a_1 + a_2)\,2\mathrm{Re}\,\varphi^{0\prime}(t)\}. \tag{4.40}$$

We have thus obtained the following boundary-value problem: it is required to find two functions $\varphi_*(z)$ and $\psi_*(z)$, holomorphic in the domain of the plate and satisfying (4.39).

If the plate occupies the exterior of contours Γ which have no inflexion points, then, according to Table 15 and (4.24), (4.25), (4.36), $\alpha_2(t)$ is always positive and vanishes nowhere on Γ.

For an unbounded plate with holes having reinforced edges, (4.39) becomes

$$a_1\varphi_*(t) + t\overline{\varphi_*'(t)} + \overline{\psi_*(t)} - a(t)[\varphi_*'(t) + \overline{\varphi_*'(t)}] = f(t), \tag{4.41}$$

where

$$f(t) = \frac{f_{**}(t)}{a_2(t)}, \qquad a(t) = \frac{itK(a_1 + a_2)\varrho}{2(\varrho \mp K)}.$$

General case of plate reinforcement by stiffening ribs

If the stiffening rib is located inside the domain of the plate and not at its edge, the magnitudes M_n, N_n, $H_{n\tau}$, X_n and Y_n in (4.1) and (4.13) represent the jumps, when crossing Γ, in the values of the corresponding moments and forces acting on the plate.

Conditions (4.2) and (4.14) become

$$\varepsilon_0 = \varepsilon_\tau^+ = \varepsilon_\tau^- \text{ on } \Gamma. \tag{4.42}$$

The superscripts plus and minus designate the magnitudes, respectively, to the left and right of Γ.

We introduce two functions $\varphi(z)$ and $\psi(z)$, piecewise-holomorphic in the domain of the middle plane of the plate and discontinuous across Γ. The approximate boundary conditions are then on Γ (by analogy with § 8 of Chapter II):

$$\begin{aligned} &iK\dot{t}\,\mathrm{Re}\left\{\dot{t}\frac{d}{ds}[a_2\overline{\varphi^+(t)} - \overline{t}\varphi'^+(t) - \psi^+(t)]\right\} + a_1\overline{\varphi^+(t)} + \\ &+ \overline{t}\varphi'^+(t) + \psi^+(t) = a_1\overline{\varphi^-(t)} + \overline{t}\varphi'^-(t) + \psi^-(t) + f_2(t), \end{aligned} \tag{4.43}$$

$$\begin{aligned} &\mathrm{Re}\left\{\dot{t}\frac{d}{ds}[a_2\overline{\varphi^+(t)} - \overline{t}\varphi'^+(t) - \psi^+(t)]\right\} = \mathrm{Re}\left\{\dot{t}\frac{d}{ds}[a_2\overline{\varphi^-(t)} - \right. \\ &\left. - \overline{t}\varphi'^-(t) - \psi^-(t)]\right\}, \end{aligned} \tag{4.44}$$

where

$$f_2(t) = -\frac{i\dot{t}A}{2D(1-\nu)}\,\mathrm{Re}\left[\dot{t}\frac{d}{ds}\left(\frac{\partial w^0}{\partial x} - i\frac{\partial w^0}{\partial y}\right)\right] - iC_1\overline{t} - C_2 \tag{4.45}$$

for the case of bending like a thin plate (w^0 is the solution at the same load, with no stiffening ribs), and

$$f_2(t) = \frac{-i}{h}\int_0^s (P_x - iP_y)\,ds + \text{const.} \tag{4.46}$$

for the two-dimensional problem. The values of a_1, a_2, K are given by (4.24) and (4.25).

§ 5. SIMPLY CONNECTED DOMAIN. BOUNDARY CONDITION IN THE MAPPED DOMAIN

We map the finite (or infinite) domain of the plate, limited by the simple closed contour Γ, on the interior (or exterior) of the unit circle γ in the plane of the complex variable $\zeta = \varrho_* e^{i\theta} = \varrho_* \sigma$ with the aid of the function $z = \omega(\zeta)$. We also write (cf. (4.38))

$$\varphi_1(\zeta) = \varphi(z) = \varphi[\omega(\zeta)], \qquad \psi_1(\zeta) = \psi(z) = \psi[\omega(\zeta)]. \tag{4.47}$$

We have on Γ:

$$\dot{t} = \pm i\sigma \frac{\omega'(\sigma)}{|\omega'(\sigma)|}, \qquad \frac{d}{ds} = \pm \frac{1}{|\omega'(\sigma)|}\frac{d}{d\theta} = \frac{\pm i\sigma}{|\omega'(\sigma)|}\frac{d}{d\sigma} \tag{4.48}$$

(signs according to Table 15).

Using (4.47) and (4.48), we transform (4.47) into

$$\alpha_4(\sigma)\left[a_1\varphi_1(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi_1'(\sigma)} + \overline{\psi_1(\sigma)}\right] - \\ - \alpha_1(\sigma)\left[\frac{\varphi'(\sigma)}{\omega'(\sigma)} + \frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\right] = f_3(\sigma) \text{ on } \gamma, \tag{4.49}$$

where

$$f_3(\sigma) = f_*[\omega(\sigma)], \qquad \alpha_1(\sigma) = i(a_1 + a_2)K\dot{t}, \\ \alpha_4(\sigma) = 2\left(1 \mp \frac{K}{\varrho}\right) = 2(1 - iK\dot{t}\,\ddot{\bar{t}}). \tag{4.50}$$

According to (4.12) $f_3(\sigma)$ is a known function of the point $\sigma = e^{i\theta}$ on γ. Due to the known indeterminacy of the functions $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ in the case of a simply-connected domain, we can in (4.35) equate to zero the constants C_1' and C_2' entering in (4.27).

We write

$$\varphi_1(\zeta) = \varphi_0(\zeta) + \varphi(\zeta), \qquad \psi_1(\zeta) = \psi_0(\zeta) + \psi(\zeta), \tag{4.51}$$

where $\varphi_0(\zeta)$ and $\psi_0(\zeta)$ represent the known solution of the problem for the plate with nonreinforced edge ($K = 0$), and the functions $\varphi(\zeta)$ and $\psi(\zeta)$, which describe the influence of the stiffening rib, are holomorphic throughout the mapped domain ζ corresponding to the plate (i. e., outside or inside the unit circle). Boundary condition (4.49) then becomes

$$\alpha_4(\sigma)\left[a_1\varphi(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)}\right] - \alpha_1(\sigma)\left[\frac{\varphi'(\sigma)}{\omega'(\sigma)} + \frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\right] = \\ = f_4(\sigma) \text{ on } \gamma, \tag{4.52}$$

where

$$f_4(\sigma) = f_*[\omega(\sigma)] + a_1(\sigma)\left[\frac{\varphi_0'(\sigma)}{\omega'(\sigma)} + \frac{\overline{\varphi_0'(\sigma)}}{\overline{\omega'(\sigma)}}\right] - a_4(\sigma) f_0[\omega(\sigma)]. \tag{4.53}$$

When no rib is provided ($K = 0$), we have $f_4(\sigma) = 0$. If the plate occupies the exterior of Γ, then $a_4(\sigma) \neq 0$ everywhere on γ, and it becomes more convenient to write (4.52) in the form

$$a_1\varphi(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} - a(\sigma)\left[\frac{\varphi'(\sigma)}{\omega'(\sigma)} + \frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\right] = f \text{ on } \gamma, \tag{4.54}$$

where

$$f = \frac{f_4(\sigma)}{2(1 - iK\dot{t}\,\dot{\bar{t}})}, \qquad a(\sigma) = \frac{i(a_1 + a_2)K\dot{t}}{2(1 - iK\dot{t}\,\dot{\bar{t}})}. \tag{4.55}$$

In order to express $a(\sigma)$ in (4.55) through $\omega(\sigma)$, we find from (4.48)

$$\dot{t}\,\dot{\bar{t}} = \frac{1}{2}\dot{t}^2\frac{d}{ds}\dot{t}^2 = \pm\frac{i\sigma^3\omega'(\sigma)}{2\overline{\omega'(\sigma)}\,|\omega'(\sigma)|}\cdot\frac{d}{d\sigma}\left[\frac{1}{\sigma^2}\cdot\frac{\overline{\omega'(\sigma)}}{\omega'(\sigma)}\right]. \tag{4.56}$$

Substituting (4.56) and (4.48) in (4.55), we obtain

$$a(\sigma) = \frac{(a_1 + a_2)\sigma\omega'(\sigma)}{2\left(1 \mp \frac{|\omega'(\sigma)|}{K}\right) - \sigma\frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}}\cdot\frac{d}{d\sigma}\cdot\frac{\overline{\omega'(\sigma)}}{\omega'(\sigma)}}. \tag{4.57}$$

Influence of a moment-free ring on the stress concentration near a circular hole in an inhomogeneously stressed plane field /153/

When Γ is a circle of radius R, we have $z = \omega(\zeta) = R\zeta$. The known functions $\varphi^0(\zeta)$ and $\psi^0(\zeta)$, which describe the stresses in an unbounded solid plate, are represented as polynomials of the m-th degree:

$$\varphi^0(\zeta) = \sum_{k=1}^{m} A_k\zeta^k, \qquad \psi^0(\zeta) = \sum_{k=0}^{m} B_k\zeta^k, \tag{4.58}$$

where A_k and B_k are generally complex coefficients.

For an unbounded plate with a circular hole of radius R, whose edge is reinforced by a thin elastic rib, we represent, in accordance with /122/, the unknown functions $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ in (4.47)

$$\left.\begin{aligned}\varphi_1(\zeta) &= \varphi^0(\zeta) + \sum_{k=1}^{m} \alpha_{-k}\zeta^{-k}\\ \psi_1(\zeta) &= \psi^0(\zeta) + \sum_{k=1}^{m+2} \beta_{-k}\zeta^{-k}\end{aligned}\right\}. \tag{4.59}$$

We assume that no external forces act on the hole, i.e., $f_1(t) = 0$ by (4.21).

Substituting (4.59) in (4.49) for $f_3(\sigma) = 0$, and equating the coefficients of equal powers of $\sigma = e^{i\theta}$, we obtain the following system of algebraic equations:

$$\left.\begin{aligned} [2 + \delta(1 + k + k\varkappa_1 - \varkappa_1)]\,\alpha_{-(k-1)} = \\ = [(\varkappa_1 - 1)\delta - 2](k+1)\bar{A}_{k+1} - 2(1+\delta)\bar{B}_{k-1} \\ \beta_{-(k+1)} = k\alpha_{-(k-1)} + k\bar{A}_{k+1} + \bar{B}_{k-1} \\ (k = 2, 3, \ldots, m+1) \end{aligned}\right\} \tag{4.60}$$

and, in addition

$$\left.\begin{aligned} (1+\delta)\beta_{-2} = (\varkappa_1\delta - 1)\bar{A}_2, \\ (1+\delta)\beta_{-1} = [(\varkappa_1 - 1)\delta - 2]A_1, \end{aligned}\right\} \tag{4.61}$$

where

$$\delta = \frac{E_1 F}{2\mu h R}, \qquad \varkappa_1 = \frac{3-\nu}{1+\nu}.$$

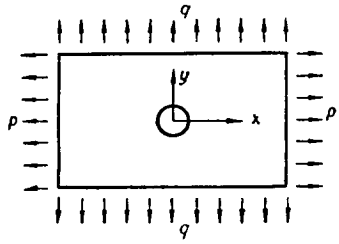


FIGURE 47.

The coefficients $\alpha_{-(k-1)}$ and $\beta_{-(k+1)}$ $(k=2, 3, \ldots, m+1)$ in (4.59) are uniquely determined from (4.60), and the problem is thus solved.

In the particular case of plane tension (Figure 47) we have

$$\varphi^0(\zeta) = \frac{p+q}{4} R\zeta, \qquad \psi^0(\zeta) = \frac{q-p}{2} R\zeta. \tag{4.62}$$

where $p = \sigma_x^{(\infty)} = \text{const.}$ and $q = \sigma_y^{(\infty)} = \text{const.}$ are the stress components at infinity. We then obtain from (4.60) and (4.61)

$$\left.\begin{aligned} \alpha_{-1} = \frac{1+\delta}{2 + (\varkappa_1 + 3)\delta}(p-q)R, \quad \beta_{-1} = \frac{(\varkappa_1 - 1)\delta - 2}{4(1+\delta)}(p+q)R \\ \beta_{-3} = \frac{2 + (1-\varkappa_1)\delta}{2 + (\varkappa_1 + 3)\delta} \cdot \frac{(p-q)R}{2} \end{aligned}\right\}. \tag{4.63}$$

All the other coefficients in (4.59) vanish. Writing $\delta = 0$, we obtain the solution for a plate with a circular hole, having a nonreinforced edge.

For $\delta = \infty$ we obtain from (4.63) the solution for the case when the edge is reinforced by an inextensible moment-free rib:

$$\left.\begin{aligned} \alpha_{-1} &= \frac{(p-q)R}{\varkappa_1+3}, \qquad \beta_{-1} = \frac{(p+q)R(\varkappa_1-1)}{4} \\ \beta_{-3} &= \frac{(p-q)R(1-\varkappa_1)}{2(\varkappa_1+3)} \end{aligned}\right\}. \tag{4.64}$$

As ought to be expected, this result differs from that obtained for a perfectly rigid washer welded into the circular hole (/118/, p. 287).

TABLE 16

According to	$k=\frac{b}{R-b}$				
	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{30}$	$\frac{1}{50}$	$\frac{1}{100}$
Equation (4.59)	1.536	1.814	2.387	2.585	2.771
/115/	1.301	1.660	2.349	2.569	2.776
/181/	1.12	1.57	2.35	—	—

It is interesting to compare the stresses found for different assumptions about the stiffening rib. Table 16 gives values of the relative stresses $\frac{\sigma_\theta}{p}$ at $\theta = \frac{\pi}{2}$, arising in the plate along the joint with the rib ($r = R$). These stresses were calculated for the case of monoaxial tension ($q=0$) of a plate reinforced by a rib of rectangular cross section whose dimensions are $2b \times 2h_1$, with $\frac{2E_1h_1}{Eh} = 1.833$, $\varkappa_1 = 2.08$, $\delta = 4.7658\,\frac{b}{R}$ for different values of $k = \frac{b}{R-b}$. For comparison, Table 16 contains also values of σ_θ taken from /115/ and /184/.

In /115/ the stiffening rib was considered as an annular plate whose stresses satisfy the equations of the two-dimensional theory of elasticity, while in /184/ the rib was assumed to be a thin bar possessing tensile and flexural rigidity.

It is seen from Table 16 that for $k \leqslant \frac{1}{20}$, the values of σ_θ, obtained for a moment-free rib, differ by less than 5% from those obtained in /115/ and /184/.

§ 6. PLANE WITH ELLIPTIC HOLE HAVING A REINFORCED EDGE. SOLUTION BY QUADRATURES

We map the domain of the plate (the exterior of an ellipse) on the region $|\zeta| > 1$, i.e., on an unbounded plane having a circular hole. The mapping function is

$$z = \omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right) \qquad (R > 0,\ 0 \leqslant m < 1). \tag{4.65}$$

The unit circle corresponds to the ellipse γ whose center is at the origin of coordinates, and whose semiaxes are

$$a = R(1+m), \qquad b = R(1-m).$$

We find from (4.65)

$$\left.\begin{aligned} \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} &= \frac{1}{\sigma}\cdot\frac{\sigma^2+m}{1-m\sigma^2}, \qquad \frac{\overline{\omega'(\sigma)}}{\omega'(\sigma)} = \frac{\sigma^2(1-m\sigma^2)}{\sigma^2-m} \\ \frac{d}{d\sigma}\left[\frac{\overline{\omega'(\sigma)}}{\omega'(\sigma)}\right] &= -\frac{2m(\sigma^4-2m\sigma^2+1)\sigma}{(\sigma^2-m)^2} \\ \omega'(\sigma) &= \frac{R(\sigma^2-m)}{\sigma^2}; \qquad \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}} = \frac{\sigma^2-m}{\sigma^2(1-m\sigma^2)} \end{aligned}\right\}. \tag{4.66}$$

Substituting (4.66) in (4.57), we obtain

$$\alpha(\sigma) = \frac{(a_1+a_2)R(\sigma^2-m)^2(m\sigma^2-1)\delta(\sigma)}{2\sigma[m\sigma^4-(1+m^2)\sigma^2-(1-m^2)\sigma^2\delta(\sigma)+m]}, \tag{4.67}$$

where

$$\delta(\sigma) = \left|\frac{K}{\omega'(\sigma)}\right| > 0$$

and

$$|\omega'(\sigma)| = R\sqrt{1+m^2-2m\cos 2\theta}. \tag{4.68}$$

The polar angle θ is measured on the ζ plane from the end of the large axis of the ellipse.

We shall henceforth assume that $\delta(\sigma)$ is a rational function of σ. Expressing $\delta(\sigma)$ in various forms, we obtain different laws of variation for the rigidity (A or E_1F) of the stiffening rib.

Consider some examples.

1) $\delta(\sigma) = \delta_0 = \text{const} > 0$. By (4.24) and (4.25) the flexural (or tensile) rigidity of the stiffening rib is in this case directly proportional to the modulus of the derivative of the mapping function

$$A = D(1-\nu)|\omega'(\sigma)|\delta_0 \quad \text{or} \quad E_1F = 2\mu h|\omega'(\sigma)|\delta_0. \tag{4.69}$$

These expressions become maximum at the end of the small axis, where $|\omega'(\sigma)| = a$, and minimum at the ends of the large axis, where $|\omega'(\sigma)| = b$. Therefore, the ratio of the extreme rigidities of the stiffening rib equals the ratio a/b of the ellipse semiaxes.

Obviously, a hole whose edge is reinforced by a rib of variable rigidity should be located in such a way that the rib has its maximum rigidity at the points of maximum stress concentration in the nonreinforced plate. Thus, when a plate with an elliptic hole is tensioned, the large axis 2a should lie in the direction of the tension. In this case the cross-sectional

area of the stiffening rib should be a/b times larger at the ends of the small axis of the ellipse than at the ends of the large axis. The variation of A and E_1F ((4.69)) is shown in Figure 48 for $m = \frac{1}{3}$, i. e., $a = 2b$.

TABLE 17

θ, deg.	$\frac{\lvert K\rvert}{R\delta_0}$ for cases		
	1	2	3
0	$1-m$	$\frac{1}{1-m}$	1
45	$\sqrt{1+m^2}$	$1/\sqrt{1+m^2}$	$\frac{\sqrt{1+m^2}}{1+\sqrt{1-m^2}}$
90	$1+m$	$\frac{1}{1+m}$	1

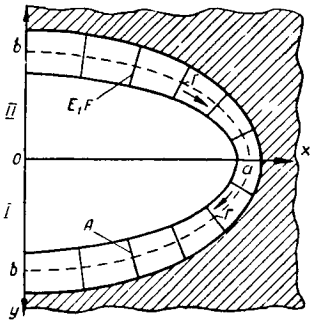


FIGURE 48.

2) $\delta(\sigma) = \sigma^2\delta_0/(\sigma^2 - m)(1 - m\sigma^2)$, where $\delta_0 = \text{const.} > 0$.
In this case

$$|K| = \frac{\sigma^2 |\omega'(\sigma)| \delta_0}{(\sigma^2 - m)(1 - m\sigma^2)} = \frac{R\delta_0}{\sqrt{1 + m^2 - 2m\cos 2\theta}}, \tag{4.70}$$

Hence, the rigidity of the rib b ecomes maximum at the ends of the large axis and minimum at the ends of the small axis. The ratio of the extremums is again equal to a/b.

3) $\delta(\sigma) = 4\sigma^2\delta_0/[(\sqrt{1-m} - i\sqrt{1+m})\sigma^2 + (\sqrt{1-m} + i\sqrt{1+m})]^2$. We have then

$$|K| = \frac{4\sigma^2 |\omega'(\sigma)| \delta_0}{[(\sqrt{1-m} - i\sqrt{1+m})\sigma^2 + (\sqrt{1-m} + i\sqrt{1+m})]^2} = \frac{R\delta_0\sqrt{1 + m^2 - 2m\cos 2\theta}}{1 - m\cos 2\theta + \sqrt{1-m^2}\sin 2\theta}, \tag{4.71}$$

where

$$\delta_0 = \text{consf} > 0.$$

The rigidities A or E_1F ((4.25) and (4.24)) are in this case equal at the ends of both axes of the ellipse.

Table 17 gives the values of $|K|/R\delta_0$ as function of the polar angle θ for these three values of $\delta(\sigma)$.

We may also assign other values to $\delta(\sigma)$, remembering that, according to (4.24) and (4.25), $\delta(\sigma)$ is a real function of constant sign.

We shall henceforth only consider the first case, when $\delta(\sigma) = \delta_0 = \text{const} > 0$.

The function $\alpha(\sigma)$ given by (4.67) will then be the boundary value of

$$\alpha(\zeta) = \frac{(a_1 + a_2) R(\zeta^2 - m)^2 (m\zeta^2 - 1)\delta_0}{2\zeta[m\zeta^4 - (1 + m^2)\zeta^2 - (1 - m^2)\zeta^2\delta + m]}. \tag{4.72}$$

To find the zeros of the function $\frac{1}{\alpha(\zeta)}$ we solve the equation

$$\beta(\zeta) = m\zeta^4 - [(1+m^2) + \delta(1-m^2)]\zeta^2 + m = 0 \tag{4.73}$$

and obtain

$$\zeta^2 = \frac{1}{2m}\left[(1+m^2) + (1-m^2)\delta_0 \pm (1-m^2)\times \sqrt{1+\delta_0^2+2\delta_0\frac{1+m^2}{1-m^2}}\right]. \tag{4.74}$$

Hence, the points ζ_l $(l = 1, 2, 3, 4)$, at which the function $\frac{1}{\alpha(\zeta)}$ becomes zero, lie on the real axis:

$$\begin{gathered}\zeta_1 = -\zeta_2 = \frac{1}{\sqrt{2m}}\times \\ \times\sqrt{(1+m^2)+(1-m^2)\delta_0+(1-m^2)\sqrt{1+\delta_0^2+2\delta_0\frac{1+m^2}{1-m^2}}}, \\ \zeta_3 = -\zeta_4 = \frac{1}{\sqrt{2m}}\times \\ \times\sqrt{(1+m^2)+(1-m^2)\delta_0-(1-m^2)\sqrt{1+\delta_0^2+2\delta_0\frac{1+m^2}{1-m^2}}}.\end{gathered} \tag{4.75}$$

We shall show that the points ζ_1, ζ_2 lie outside the unit circle (γ).

In fact, we obtain from (4.75)

$$\zeta_1^2 = \zeta_2^2 > \frac{1}{2m}[(1+m^2)+(1-m^2)\delta_0+(1-m^2)(1+\delta_0)] = \frac{1+(1-m^2)\delta_0}{m} > 1.$$

Applying Viéte's theorem to (4.73), we find that $\zeta_1^2\cdot\zeta_3^2 = 1$. Thus,

$$\zeta_3 = \frac{1}{\zeta_1}, \qquad \zeta_4 = -\frac{1}{\zeta_1}. \tag{4.76}$$

This means that the points ζ_3 and ζ_4 lie inside the unit circle (γ).

Multiplying (4.54) by $\frac{1}{2\pi i}\cdot\frac{d\sigma}{\sigma-\zeta}$ and integrating along γ for $|\zeta|>1$, we obtain

$$\begin{gathered}\frac{a_1}{2\pi i}\int_\gamma \varphi(\sigma)\frac{d\sigma}{\sigma-\zeta} + \frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)}\frac{d\sigma}{\sigma-\zeta} + \frac{1}{2\pi i}\int_\gamma \overline{\psi(\sigma)}\frac{d\sigma}{\sigma-\zeta} - \\ -\frac{1}{2\pi i}\int_\gamma \alpha(\sigma)\left[\frac{\varphi'(\sigma)}{\omega'(\sigma)} + \frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\right]\frac{d\sigma}{\sigma-\zeta} = \frac{1}{2\pi i}\int_\gamma \frac{f d\sigma}{\sigma-\zeta}.\end{gathered} \tag{4.77}$$

The functions $\varphi(\zeta)$ and $\psi(\zeta)$ are holomorphic outside γ, including the point $\zeta=\infty$, where $\varphi(\infty)=0$, $\psi(\infty) =$ const.

holomorphic everywhere outside γ, except at the points $\zeta=\zeta_1$ and $\zeta=-\zeta_1=\zeta_2$, where it has poles of the first order. Its principal parts are, respectively,

$$G_1(\zeta)=-\frac{C_1}{\zeta-\zeta_1}, \qquad G_2(\zeta)=-\frac{C_2}{\zeta-\zeta_2}, \tag{4.83}$$

where

$$\left.\begin{aligned} C_1 &= \frac{(a_1+a_2)(\zeta_1^2-m)(1-m\zeta_1^2)\zeta_1^2\delta_0}{4(\zeta_1^4-1)}\varphi'(\zeta_1) \\ C_2 &= \frac{(a_1+a_2)(\zeta_1^2-m)(1-m\zeta_1^2)\zeta_1^2\delta_0}{4(\zeta_1^4-1)}\varphi'(-\zeta_1) \end{aligned}\right\}. \tag{4.84}$$

Thus,

$$\frac{1}{2\pi i}\int_\gamma \alpha(\sigma)\frac{\varphi'(\sigma)}{\omega'(\sigma)}\cdot\frac{d\sigma}{\sigma-\zeta}=\frac{(a_1+a_2)(\zeta^2-m)(1-m\zeta^2)\zeta\delta_0}{2(\zeta^2-\zeta_1^2)(\zeta^2-\zeta_3^2)}\varphi'(\zeta)-\frac{C_1}{\zeta-\zeta_1}-\frac{C_2}{\zeta-\zeta_2}. \tag{4.85}$$

Substituting (4.78), (4.81), and (4.85) in (4.77), we obtain

$$\varphi(\zeta)+P(\zeta)\varphi'(\zeta)=Q(\zeta), \tag{4.86}$$

where

$$\left.\begin{aligned} P(\zeta) &= \frac{(a_1+a_2)\delta_0}{2a_1}\cdot\frac{\zeta(\zeta^2-m)(1-m\zeta^2)}{m\zeta^4-(1+m^2)\zeta^2-(1-m^2)\zeta^2\delta_0+m} \\ Q(\zeta) &= \frac{1}{a_1}\left(\sum_{k=1}^{4}\frac{C_k}{\zeta-\zeta_k}-\frac{1}{2\pi i}\int_\gamma\frac{f\,d\sigma}{\sigma-\zeta}\right) \end{aligned}\right\}. \tag{4.87}$$

We have thus obtained an ordinary linear differential equation of the first order in the complex domain /1, 25/ from which the function $\varphi(\zeta)$, holomorphic everywhere outside γ, can be determined.

Integrating (4.86), we find

$$\varphi(\zeta)=e^{-\int\frac{d\zeta}{P(\zeta)}}\left[C_*+\int\frac{Q(\zeta)}{P(\zeta)}e^{\int\frac{d\zeta}{P(\zeta)}}d\zeta\right], \tag{4.88}$$

where C_* is a constant.

From the condition

$$\lim_{\zeta\to\infty}\varphi(\zeta)=0$$

By Cauchy's integral formula,

$$\left.\begin{aligned} &\frac{1}{2\pi i}\int_\gamma \frac{\varphi(\sigma)\,d\sigma}{\sigma-\zeta} = -\varphi(\zeta), \qquad \frac{1}{2\pi i}\int_\gamma \frac{\overline{\psi(\sigma)}\,d\sigma}{\sigma-\zeta} = 0 \\ &\frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)}\frac{d\sigma}{\sigma-\zeta} = \frac{1}{2\pi i}\int_\gamma \frac{1}{\sigma}\frac{\sigma^2+m}{1-m\sigma^2}\overline{\varphi'}\left(\frac{1}{\sigma}\right)\frac{d\sigma}{\sigma-\zeta} = 0 \end{aligned}\right\}. \tag{4.78}$$

To evaluate the integral

$$\frac{1}{2\pi i}\int_\gamma \alpha(\sigma)\frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\cdot\frac{d\sigma}{\sigma-\zeta} = \frac{1}{2\pi i}\int_\gamma \frac{-(a_1+a_2)\delta_0(\sigma^2-m)^2\,\overline{\varphi}'\left(\frac{1}{\sigma}\right)}{2\sigma(\sigma^2-\zeta_1^2)(\sigma^2-\zeta_3^2)}\cdot\frac{d\sigma}{\sigma-\zeta}$$

we note that the integrand represents the boundary value of the function

$$-\frac{(a_1+a_2)(\zeta^2-m)^2\delta_0}{2\zeta(\zeta^2-\zeta_1^2)(\zeta^2-\zeta_3^2)}\overline{\varphi}'\left(\frac{1}{\zeta}\right),$$

holomorphic everywhere inside γ except at the points $\zeta=\zeta_3$ and $\zeta=\zeta_4$, where it has poles of the first order. Its principal parts are, respectively,

$$G_3(\zeta)=\frac{C_3}{\zeta-\zeta_3}, \qquad G_4(\zeta)=\frac{C_4}{\zeta+\zeta_3}. \tag{4.79}$$

Here

$$\left.\begin{aligned} C_3 &= -\frac{(a_1+a_2)(1-m\zeta_1^2)^2\delta_0}{4(1-\zeta_1^4)}\overline{\varphi}'(\zeta_1) \\ C_4 &= -\frac{(a_1+a_2)(1-m\zeta_1^2)^2\delta_0}{4(1-\zeta_1^4)}\overline{\varphi}'(-\zeta_1) \end{aligned}\right\}. \tag{4.80}$$

Thus,

$$\frac{1}{2\pi i}\int_\gamma \alpha(\sigma)\frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\cdot\frac{d\sigma}{\sigma-\zeta} = -\frac{C_3}{\zeta-\zeta_3}-\frac{C_4}{\zeta-\zeta_4}. \tag{4.81}$$

The integrand in

$$\frac{1}{2\pi i}\int_\gamma \alpha(\sigma)\frac{\varphi'(\sigma)}{\omega'(\sigma)}\cdot\frac{d\sigma}{\sigma-\zeta} = \frac{1}{2\pi i}\int_\gamma \frac{-(a_1+a_2)(\sigma^2-m)(1-m\sigma^2)\sigma\delta_0}{2(\sigma^2-\zeta_1^2)(\sigma^2-\zeta_3^2)}\varphi'(\sigma)\frac{d\sigma}{\sigma-\zeta}$$

is the boundary value of the function

$$-\frac{(a_1+a_2)(\zeta^2-m)(1-m\zeta^2)\zeta\delta_0}{2(\zeta^2-\zeta_1^2)(\zeta^2-\zeta_3^2)}\varphi'(\zeta), \tag{4.82}$$

where

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\varphi'(\sigma)\frac{d\sigma}{\sigma-\zeta}=\frac{1}{2\pi i}\int_{\gamma}\sigma\frac{1+m\sigma^2}{\sigma^2-m}\varphi'(\sigma)\frac{d\sigma}{\sigma-\zeta}=-\zeta\frac{1+m\zeta^2}{\zeta^2-m}\varphi'(\zeta). \qquad (4.97)$$

The integrand in

$$\frac{1}{2\pi i}\int_{\gamma}\overline{\alpha(\sigma)}\frac{\varphi'(\sigma)}{\omega'(\sigma)}\cdot\frac{d\sigma}{\sigma-\zeta}=\frac{1}{2\pi i}\int_{\gamma}-\frac{(a_1+a_2)(1-m\sigma^2)^2\sigma\delta_0\varphi'(\sigma)}{2(1-\zeta_1\sigma)(1-\zeta_2\sigma)(1-\zeta_3\sigma)(1-\zeta_4\sigma)}\cdot\frac{d\sigma}{\sigma-\zeta}$$

represents the boundary value of the function

$$\frac{(a_1+a_2)(1-m\zeta^2)^2\zeta\varphi'(\zeta)\zeta_1^2\delta_0}{2(1-\zeta_1^2\zeta^2)(\zeta^2-\zeta_1^2)},$$

holomorphic everywhere outside γ except at the points $\zeta=\frac{1}{\zeta_3}=\zeta_1$ and $\zeta=-\zeta_1$, where it has poles of the first order; its principal parts are, respectively,

$$G_3^*(\zeta)=-\frac{C_3^*}{\zeta-\zeta_1},\qquad G_4^*(\zeta)=-\frac{C_4^*}{\zeta+\zeta_1}, \qquad (4.98)$$

where

$$\begin{aligned}C_3^*&=-\frac{(a_1+a_2)(1-m\zeta_1^2)^2\zeta_1^2\delta_0}{4(1-\zeta_1^4)}\varphi'(\zeta_1),\\ C_4^*&=-\frac{(a_1+a_2)(1-m\zeta_1^2)^2\zeta_1^2\delta_0}{4(1-\zeta_1^4)}\varphi'(-\zeta_1).\end{aligned} \qquad (4.99)$$

Thus,

$$\frac{1}{2\pi i}\int_{\gamma}\overline{\alpha(\sigma)}\frac{\varphi'(\sigma)}{\omega'(\sigma)}\cdot\frac{d\sigma}{\sigma-\zeta}=\frac{(a_1+a_2)(1-m\zeta^2)^2\zeta\zeta_1^2\delta_0}{2(1-\zeta_1^2\zeta^2)(\zeta_1^2-\zeta^2)}\varphi'(\zeta)-\frac{C_3^*}{\zeta-\zeta_1}-\frac{C_4^*}{\zeta+\zeta_1}. \qquad (4.100)$$

The integrand in

$$\frac{1}{2\pi i}\int_{\gamma}\overline{\alpha(\sigma)}\frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\cdot\frac{d\sigma}{\sigma-\zeta}=\frac{1}{2\pi i}\int_{\gamma}\frac{-(a_1+a_2)(1-m\sigma^2)(\sigma^2-m)\delta_0}{2\sigma(1-\zeta_1^2\sigma^2)(1-\zeta_3^2\sigma^2)}\bar{\varphi}'\left(\frac{1}{\sigma}\right)\frac{d\sigma}{\sigma-\zeta}$$

is the boundary value of the function

$$-\frac{(a_1+a_2)(1-m\zeta^2)(\zeta^2-m)}{2\zeta(1-\zeta_1^2\zeta^2)(1-\zeta_3^2\zeta^2)}\delta_0\bar{\varphi}'\left(\frac{1}{\zeta}\right),$$

we find $C_* = 0$, whence

$$\varphi(\zeta) = e^{-\int \frac{dq}{P(\zeta)}} \int \frac{Q(\zeta)}{P(\zeta)} e^{\int \frac{dq}{P(\zeta)}} d\zeta. \tag{4.89}$$

Using (4.87), we find:

$$\int \frac{d\zeta}{P(\zeta)} = \frac{2a_1}{(a_1+a_2)\delta_0} \int \frac{m\zeta^4 - (1+m^2)\zeta^2 - (1-m^2)\zeta^2\delta_0 + m}{\zeta(\zeta^2-m)(1-m\zeta^2)} d\zeta =$$

$$= \frac{2a_1}{(a_1+a_2)\delta_0} \ln \left[\frac{(1-m\zeta^2)^{\frac{\delta_0}{2}}}{\zeta(\zeta^2-m)^{\frac{\delta_0}{2}}} \right] = \ln \left[\frac{1-m\zeta^2}{(\zeta^2-m)\zeta^{\frac{2}{\delta_0}}} \right]^{\frac{a_1}{a_1+a_2}}. \tag{4.90}$$

Hence,

$$e^{\int \frac{dq}{P(\zeta)}} = \left[\frac{1-m\zeta^2}{(\zeta^2-m)\zeta^{\frac{2}{\delta_0}}} \right]^{\frac{a_1}{a_1+a_2}}. \tag{4.91}$$

Substitution of (4.91) in (4.88) yields

$$\varphi(\zeta) = \left[\frac{\zeta^{\frac{2}{\delta_0}}(\zeta^2-m)}{1-m\zeta^2} \right]^{\frac{a_1}{a_1+a_2}} \int \frac{Q(\zeta)}{P(\zeta)} \left[\frac{1-m\zeta^2}{\zeta^{\frac{2}{\delta_0}}(\zeta^2-m)} \right]^{\frac{a_1}{a_1+a_2}} d\zeta, \tag{4.92}$$

where

$$\frac{Q(\zeta)}{P(\zeta)} = \frac{2(\zeta-\zeta_1)(\zeta-\zeta_2)(\zeta-\zeta_3)(\zeta-\zeta_4)}{(a_1+a_2)(\zeta^2-m)(1-m\zeta^2)\zeta\delta_0} \left[\sum_{k=1}^{4} \frac{C_k}{\zeta-\zeta_k} - \frac{1}{2\pi i} \int_\gamma \frac{f d\sigma}{\sigma-\zeta} \right]. \tag{4.93}$$

From (4.92), we can obtain $\varphi(\zeta)$, proceeding from the boundary condition complex conjugate to (4.54),

$$a_1 \overline{\varphi(\sigma)} + \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \varphi'(\sigma) + \psi(\sigma) - \overline{\alpha(\sigma)} \left[\frac{\varphi'(\sigma)}{\omega'(\sigma)} + \frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}} \right] = \bar{f}. \tag{4.94}$$

We have

$$\psi(\zeta) = -\frac{1}{2\pi i} \int_\gamma \frac{\psi(\sigma) d\sigma}{\sigma-\zeta} + \psi(\infty). \tag{4.95}$$

Substituting in (4.95) for $\psi(\sigma)$ from (4.94), we obtain

$$\psi(\zeta) = \frac{1}{2\pi i} \int_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \varphi'(\sigma) \frac{d\sigma}{\sigma-\zeta} - \frac{1}{2\pi i} \int_\gamma \overline{\alpha(\sigma)} \left[\frac{\varphi'(\sigma)}{\omega'(\sigma)} + \right.$$

$$\left. + \frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}} \right] \frac{d\sigma}{\sigma-\zeta} - \frac{1}{2\pi i} \int_\gamma \frac{\bar{f} d\sigma}{\sigma-\zeta} + \psi(\infty) \tag{4.96}$$

holomorphic everywhere inside γ except at the points $\zeta = \frac{1}{\zeta_1} = \zeta_3$ and $\zeta = -\frac{1}{\zeta_1} = -\zeta_3$, where it has poles of the first order. Its principal parts are, respectively,

$$G_1^*(\zeta) = \frac{C_1^*}{\zeta - \zeta_3}, \qquad G_2^*(\zeta) = \frac{C_2^*}{\zeta + \zeta_3}, \tag{4.101}$$

where

$$\left.\begin{aligned} C_1^* &= \frac{(a_1 + a_2)(\zeta_1^2 - m)(1 - m\zeta_1^2)\delta_0}{4(\zeta_1^4 - 1)}\,\overline{\varphi}'(\zeta_1) \\ C_2^* &= \frac{(a_1 + a_2)(\zeta_2^2 - m)(1 - m\zeta_2^2)\delta_0}{4(\zeta_1^4 - 1)}\,\overline{\varphi}'(-\zeta_1) \end{aligned}\right\}. \tag{4.102}$$

Thus,

$$\frac{1}{2\pi i}\int_\gamma \overline{\alpha(\sigma)}\,\frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\cdot\frac{d\sigma}{\sigma - \zeta} = -\frac{C_1^*}{\zeta - \zeta_3} - \frac{C_2^*}{\zeta + \zeta_3}. \tag{4.103}$$

Substituting (4.97), (4.100), and (4.103) in (4.96), we obtain finally

$$\psi(\zeta) = -\zeta\frac{1 + m\zeta^2}{\zeta^2 - m}\varphi'(\zeta) - \frac{(a_1 + a_2)(1 - m\zeta^2)^2\delta_0\zeta\varphi'(\zeta)}{2[m\zeta^4 - (1 + m^2)\zeta^2 - (1 - m^2)\delta_0\zeta^2 + m]} + \\ + \sum_{k=1}^{4}\frac{C_k^*}{\zeta - \zeta_k} - \frac{1}{2\pi i}\int_\gamma \frac{\overline{f}\,d\sigma}{\sigma - \zeta} + \psi(\infty). \tag{4.104}$$

Here,

$$\psi(\infty) = \frac{1}{2\pi i}\int_\gamma \frac{\psi(\sigma)\,d\sigma}{\sigma}, \tag{4.105}$$

where by (4.94):

$$\psi(\sigma) = \overline{f} - a_1\overline{\varphi(\sigma)} - \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\varphi'(\sigma) + \overline{\alpha(\sigma)}\left[\frac{\varphi'(\sigma)}{\omega'(\sigma)} + \frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\right].$$

Substituting this in (4.105), and noting that by (4.78), (4.97), (4.100), and (4.103),

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{\varphi(\sigma)}\,d\sigma}{\sigma} = 0, \qquad \frac{1}{2\pi i}\int_\gamma \frac{1 + m\sigma^2}{\sigma^2 - m}\varphi'(\sigma)\,d\sigma = 0,$$

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{\alpha(\sigma)}\,\varphi'(\sigma)}{\sigma\omega'(\sigma)}\,d\sigma = G_3^*(0) + G_4^*(0) = (C_3^* - C_4^*)\frac{1}{\zeta_1},$$

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\alpha(\sigma)}\,\overline{\varphi'(\sigma)}}{\sigma\,\overline{\omega'(\sigma)}}\,d\sigma=-G_1^*(0)-G_2^*(0)=(C_1^*-C_2^*)\zeta_1,$$

we obtain

$$\psi(\infty)=\frac{1}{2\pi i}\int_{\gamma}\frac{\bar{f}d\sigma}{\sigma}+(\bar{C}_3-\bar{C}_4)\zeta_1+(\bar{C}_1-\bar{C}_2)\frac{1}{\zeta_1}.$$

It can be shown by direct substitution that the boundary conditions (4.54) or (4.94) are satisfied by the functions $\varphi(\zeta)$ and $\psi(\zeta)$, given, respectively, by (4.92) and (4.104), for any values of C_i and C_i^* $(i=1, 2, 3, 4)$. The latter are in accordance with (4.80), (4.84), (4.99), and (4.102), expressed through the still unknown magnitudes $\varphi'(\zeta_1)$ and $\varphi'(-\zeta_1)$.

Since the root ζ_1 is real, we have

$$\overline{\varphi'(\zeta_1)}=\overline{\varphi'(\zeta_1)},\qquad \overline{\varphi'(-\zeta_1)}=\overline{\varphi'(-\zeta_1)}. \tag{4.105a}$$

Hence, to determine the constants

$$C_1=\bar{C}_1^*\zeta_1^2,\quad C_2=\bar{C}_2^*\zeta_1^2,\quad C_3=\bar{C}_3^*\frac{1}{\zeta_1^2},\quad C_4=\bar{C}_4^*\frac{1}{\zeta_1^2} \tag{4.106}$$

we require only the two magnitudes

$$\varphi'(\zeta_1),\qquad \varphi'(-\zeta_1). \tag{4.107}$$

For this, we must first show that these quantities represent the values of $\varphi'(\zeta)$ at the points ζ_1 and $-\zeta_1$.

Differentiating (4.92), we find

$$\varphi'(\zeta)=\frac{1}{P(\zeta)}[Q(\zeta)-\varphi(\zeta)]. \tag{4.108}$$

The same result can also be obtained directly from (4.86) [sic].

Setting in (4.108) $\zeta=\zeta_1$ and assuming that the function $\varphi(\zeta)$ is holomorphic everywhere outside γ, we find from (4.93) [since $P(\zeta_1)\to\infty$]

$$\varphi'(\zeta_1)=-\frac{2C_1(\zeta_1-\zeta_2)(\zeta_1-\zeta_3)(\zeta_1-\zeta_4)}{(a_1+a_2)(\zeta_1^2-m)(1-m\zeta_1^2)\zeta_1\delta_0}, \tag{4.109}$$

which, by (4.84), is satisfied identically. A similar identity is obtained by setting in (4.108) $\zeta=\zeta_2=-\zeta_1$.

The unknown constants (4.107) will be determined from the condition that the solution of (4.86) or (4.108) be holomorphic. Cauchy's theorem /25/ states that the function $\varphi(\zeta)$ is holomorphic everywhere outside γ if the right-hand side of (4.108) is holomorphic everywhere outside γ. The right-hand side of (4.108) may have singularities outside γ at the points

$$\zeta_5=\frac{1}{\sqrt{m}},\qquad \zeta_6=-\frac{1}{\sqrt{m}}. \tag{4.110}$$

where

$$\left.\begin{aligned} a_{jk} &= \frac{1}{\zeta_{4+j}-\zeta_k} - \omega_k(\zeta_{4+j}) \qquad (j, k = 1, 2) \\ a_{jk} &= \frac{1-m\zeta_1^2}{\zeta_1^2(\zeta_1^2-m)}\left[\frac{1}{\zeta_{4+j}-\zeta_k} - \omega_k(\zeta_{4+j})\right] \qquad \begin{pmatrix} j=1,2 \\ k=3,4 \end{pmatrix} \\ a_{j0} &= \frac{1}{2\pi i}\int \frac{f\,d\sigma}{\sigma-\zeta_{4+j}} + \omega_0(\zeta_{4+j}) \qquad (j=1,2) \end{aligned}\right\}. \tag{4.119}$$

All the coefficients of the unknowns in (4.118) are real, while the free terms are generally complex.

Rewriting (4.118) in complex conjugate magnitudes, we obtain

$$\left.\begin{aligned} a_{11}\overline{C}_1 + a_{12}\overline{C}_2 + a_{13}C_1 + a_{14}C_2 &= \overline{a}_{10} \\ a_{21}\overline{C}_1 + a_{22}\overline{C}_2 + a_{23}C_1 + a_{24}C_2 &= \overline{a}_{20} \end{aligned}\right\}. \tag{4.120}$$

Solving system (4.118), (4.120), we find

$$\left.\begin{aligned} \operatorname{Im} C_1 &= \frac{1}{\Delta_1}\operatorname{Im}[a_{10}(a_{22}-a_{24}) - a_{20}(a_{12}-a_{14})] \\ \operatorname{Im} C_2 &= -\frac{1}{\Delta_1}\operatorname{Im}[a_{10}(a_{21}-a_{23}) - a_{20}(a_{11}-a_{13})] \\ \operatorname{Re} C_1 &= \frac{1}{\Delta_2}\operatorname{Re}[a_{10}(a_{22}+a_{24}) - a_{20}(a_{12}+a_{14})] \\ \operatorname{Re} C_2 &= -\frac{1}{\Delta_2}\operatorname{Re}[a_{10}(a_{21}+a_{23}) - a_{20}(a_{11}+a_{13})] \end{aligned}\right\}, \tag{4.121}$$

where

$$\left.\begin{aligned} \Delta_1 &= (a_{11}-a_{13})(a_{22}-a_{24}) - (a_{21}-a_{23})(a_{12}-a_{14}) \\ \Delta_2 &= (a_{11}+a_{13})(a_{22}+a_{24}) - (a_{21}+a_{23})(a_{12}+a_{14}) \end{aligned}\right\}. \tag{4.122}$$

From the coefficients C_1 and C_2 given by (4.121), we find C_3 and C_4 by (4.117) and C_j^* $(j = 1, 2, 3, 4)$ by (4.106); the functions $\varphi(\zeta)$ ((4.92)) and $\psi(\zeta)$ ((4.104)) are therefore completely determined and the problem is solved.

We can solve this problem similarly also for $\delta(\sigma) \neq$ const., but in this case (4.73) will no longer be biquadratic. Determination of its roots is then more difficult, so that the solution becomes more involved.

The solution given here can also be generalized to apply when $z = \omega(\zeta)$ ((4.65)) is any rational function.

Infinite plate having a circular hole with reinforced edge

Setting $m = 0$ in the above formulas we obtain the solution for an infinite plate reinforced by a rib of constant rigidity around a circular hole of radius R.

at which the function $P(\zeta)$ (4.87) in the denominator of (4.108) becomes zero. We therefore require that the numerator of (4.108) also vanish at points (4.110), i.e.,

$$Q(\zeta_5) - \varphi(\zeta_5) = 0, \qquad Q(\zeta_6) - \varphi(\zeta_6) = 0. \tag{4.111}$$

By virtue of (4.84) we can write (4.92) as follows:

$$\varphi(\zeta) = \Omega(\zeta) \int \frac{1}{a_1 P(\zeta)\,\Omega(\zeta)} \left[\sum_{k=1}^{4} \frac{C_k}{\zeta - \zeta_k} - \frac{1}{2\pi i} \int_{\gamma} \frac{f d\sigma}{\sigma - \zeta} \right] d\zeta \tag{4.112}$$

where

$$\Omega(\zeta) = \left[\frac{\zeta^{\delta_0} (\zeta^2 - m)}{1 - m\zeta^2} \right]^{\frac{a_1}{a_1 + a_4}}. \tag{4.113}$$

Expression (4.112) can also be written in the following form:

$$a_1 \varphi(\zeta) = \omega_0(\zeta) + \sum_{k=1}^{4} C_k \omega_k(\zeta), \tag{4.114}$$

where

$$\left.\begin{aligned} \omega_0(\zeta) &= -\Omega(\zeta) \int \frac{1}{P(\zeta)\,\Omega(\zeta)} \left[\frac{1}{2\pi i} \int_{\gamma} \frac{f d\sigma}{\sigma - \zeta} \right] d\zeta, \\ \omega_k(\zeta) &= \Omega(\zeta) \int \frac{1}{P(\zeta)\,\Omega(\zeta)} \cdot \frac{d\zeta}{\zeta - \zeta_k} \qquad (k = 1, 2, 3, 4). \end{aligned}\right\} \tag{4.115}$$

The functions $\omega_0(\zeta)$ and $\omega_k(\zeta)$ are known. Substituting (4.114) and (4.87) in (4.111), we find

$$\left.\begin{aligned} \sum_{k=1}^{4} C_k \left[\frac{1}{\zeta_5 - \zeta_k} - \omega_k(\zeta_5) \right] &= \frac{1}{2\pi i} \int_{\gamma} \frac{f d\sigma}{\sigma - \zeta_5} + \omega_0(\zeta_5) \\ \sum_{k=1}^{4} C_k \left[\frac{1}{\zeta_6 - \zeta_k} - \omega_k(\zeta_6) \right] &= \frac{1}{2\pi i} \int_{\gamma} \frac{f d\sigma}{\sigma - \zeta_6} + \omega_0(\zeta_6) \end{aligned}\right\}. \tag{4.116}$$

Comparing (4.80) with (4.84) and noting (4.105a), we find

$$\frac{C_3}{\overline{C}_1} = \frac{C_4}{\overline{C}_2} = \frac{1 - m\zeta_1^2}{(\zeta_1^2 - m)\,\zeta_1^2}. \tag{4.117}$$

Eliminating the constants C_3 and C_4 from (4.116) with the aid of (4.117), we obtain

$$\left.\begin{aligned} a_{11} C_1 + a_{12} C_2 + a_{13} \overline{C}_1 + a_{14} \overline{C}_2 &= a_{10} \\ a_{21} C_1 + a_{22} C_2 + a_{23} \overline{C}_1 + a_{24} \overline{C}_2 &= a_{20} \end{aligned}\right\}, \tag{4.118}$$

In this case

$$\delta_0 = \frac{K}{R}, \qquad \zeta_1 = \infty, \qquad \zeta_2 = 0, \qquad \varphi'(\zeta_1) = 0,$$

$$C_i^* = C_i = 0 \qquad (i = 1, 2, 3, 4),$$

$$Q(\zeta) = -\frac{1}{a_1}\cdot\frac{1}{2\pi i}\int_\gamma \frac{f d\sigma}{\sigma - \zeta}, \qquad P(\zeta) = -\frac{(a_1 + a_2)\delta_0}{2a_1(1 + \delta_0)}\zeta,$$

and, according to (4.92) and (4.104),

$$\left.\begin{aligned} \varphi(\zeta) &= \frac{\nu_2}{a_1}\zeta^{\nu_2}\int A_*(\zeta)\zeta^{-1-\nu_2}d\zeta \\ \psi(\zeta) &= -\frac{\nu_1}{\zeta}\varphi'(\zeta) - \frac{1}{2\pi i}\int_\gamma \frac{\bar{f} d\sigma}{\sigma - \zeta} + \psi(\infty) \end{aligned}\right\}, \tag{4.123}$$

where

$$\nu_1 = \frac{2(1 + \delta_0) - (a_1 + a_2)\delta_0}{2(1 + \delta_0)}, \qquad \nu_2 = \frac{2a_1(1 + \delta_0)}{(a_1 + a_2)\delta_0} > 0,$$

$$A_*(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{f d\sigma}{\sigma - \zeta}, \quad \psi(\infty) = \frac{1}{2\pi i}\int_\gamma \frac{\bar{f} d\sigma}{\sigma}.$$

Representing the known function $A_*(\zeta)$ outside γ as a Fourier series

$$A_*(\zeta) = \sum_{k=1}^{\infty} A_k \zeta^{-k},$$

we obtain from (4.123)

$$\varphi(\zeta) = -\frac{\nu_2}{a_1}\sum_{k=1}^{\infty}\frac{A_k}{k + \nu_2}\zeta^{-k}. \tag{4.124}$$

It is easily proved that (4.124) coincides with $\varphi_1(\zeta)$ in (4.59).

We can similarly obtain the solution for a circular hole whose edge is reinforced by a thin rib of variable rigidity.

§ 7. EMBOSSED PLATE. BOUNDARY CONDITIONS

Plates are frequently embossed in various shapes in order to increase the rigidity. Figure 49 shows section and plan of a plate with a single embossing. The plate carries an arbitrary transverse load.

Such a structure consists actually of two plates, I and II, linked by an annular stiffening rib. For the sake of generality we shall assume that the thicknesses h_1 and h_2 of these plates differ. The distance between the middle planes of the plates is $2h_0$. The axis of the stiffening rib is denoted

by Γ. The interaction between plates and rib is shown schematically in Figure 50. The xy-plane contains the curve Γ inside which the origin is located. The Z-axis points downward. Let S be the cylindrical surface perpendicular to the xy-plane which passes through Γ, and let Γ_1 and Γ_2 be the curves along which this surface intersects the middle planes of plates I and II.

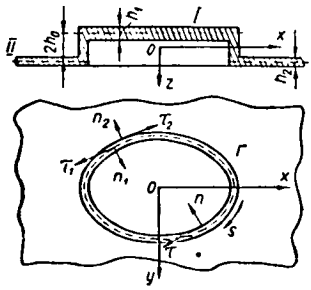


FIGURE 49.

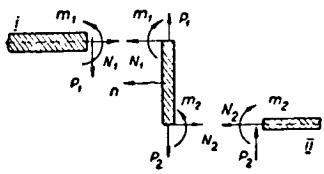


FIGURE 50.

We assume the rib to be so thin that its resistance to twisting and to bending in the xy-plane can be neglected (cf. § 1 of this chapter). In other words, we assume that the rib cross sections are not subjected to twisting, nor to bending moments whose vector is parallel to the Z-axis. We also assume that the interaction between the plates and the stiffening rib takes place on the surface S.

The plates adjoining the rib act on the latter with the following forces (per unit length of Γ):

a) The forces N_* and T_* in the xy-plane;

$$N_* + iT_* = (N_1 - N_2) - i(T_1 - T_2). \tag{4.125}$$

Here N_k and $T_k (k = 1, 2)$ are, respectively, the projections on the normal n_k and on the tangent τ_k to Γ_k of the principal vector of the stresses acting in plates I and II along Γ_k (per unit length); N_* and Γ_k are, respectively, the resultant projections on n and τ of the forces acting on unit length of the rib axis Γ. The normal n to Γ is directed toward the center of curvature, while the direction of the tangent τ is shown in Figure 49. The forces T_1 and T_2, which are normal to the plane of the section, are not shown in Figure 50; their directions are, respectively, away from, and toward the reader.

b) The shearing forces p normal to the xy-plane and the bending moments m:

$$\left.\begin{aligned} p &= (p_2 - p_1) + h_0 \frac{d}{ds}(T_2 + T_1) \\ m &= (m_2 - m_1) + h_0 (N_1 + N_2) \end{aligned}\right\}, \tag{4.126}$$

where p_k and $m_k (k = 1, 2)$ are, respectively, the shearing force and the bending moment acting on Γ_k.

The external load applied directly to the rib can be referred either to plate I or to plate II on T_*.

Under the action of loads (4.125) and (4.126) the annular stiffening rib is deformed together with the plates adjoining it. We approximate the compatibility conditions for rib and plates on S by equating the respective strains (ε_k) and (ε_0) of those fibers of the plates and rib which are equidistant from the xy-plane.

$$\varepsilon_1 = \varepsilon_0 \quad \text{for all} \quad -\left(h_0 + \frac{h_1}{2}\right) \leqslant Z \leqslant -\left(h_0 - \frac{h_1}{2}\right), \tag{4.127}$$

$$\varepsilon_2 = \varepsilon_0 \quad \text{for all} \quad \left(h_0 - \frac{h_2}{2}\right) \leqslant Z \leqslant \left(h_0 + \frac{h_2}{2}\right). \tag{4.128}$$

According to (4.16) and (4.7), where Z has been replaced by $Z \pm h_0$, we have

$$\varepsilon_k = -[Z - (-1)^k h_0] \operatorname{Re}\left[i \frac{d}{ds}\left(\frac{\partial w_k}{\partial x} - i\frac{\partial w_k}{\partial y}\right)\right] + \operatorname{Re}\left[i \frac{d}{ds}(u_k + iv_k)\right] \qquad (k = 1, 2). \tag{4.129}$$

In addition, by (4.17) and (4.5):

$$\varepsilon_0 = \frac{N}{E_1 F} + \frac{L_n}{A} Z. \tag{4.130}$$

Substituting (4.129) and (4.130) in (4.127) and (4.128), we obtain

$$\begin{gathered}[(-1)^k h_0 - Z] \operatorname{Re}\left[i \frac{d}{ds}\left(\frac{\partial w_k}{\partial x} - i\frac{\partial w_k}{\partial y}\right)\right] + \operatorname{Re}\left[i \frac{d}{ds}(u_k + iv_k)\right] = \\ = \frac{L_n}{A} Z + \frac{N}{E_1 F} \qquad (k = 1, 2), \\ \left[-\frac{h_k}{2} + (-1)^k h_0\right] \leqslant Z \leqslant \left[\frac{h_k}{2} + (-1)^k h_0\right],\end{gathered} \tag{4.131}$$

where u_k, v_k, w_k are the displacement components, respectively, in the x, y, Z. directions. Since (4.131) must be fulfilled for any Z in these intervals, we must have

$$(-1)^k h_0 \operatorname{Re}\left[i \frac{d}{ds}\left(\frac{\partial w_k}{\partial x} - i\frac{\partial w_k}{\partial y}\right)\right] + \operatorname{Re}\left[i \frac{d}{ds}(u_k + iv_k)\right] = \frac{N}{E_1 F}, \tag{4.132}$$

$$-\operatorname{Re}\left[i \frac{d}{ds}\left(\frac{\partial w_k}{\partial x} - i\frac{\partial w_k}{\partial y}\right)\right] = \frac{L_n}{A} \qquad (k = 1, 2). \tag{4.133}$$

Subtracting one from another the equations (4.133) corresponding to $k = 1$ and $k = 2$, respectively, we obtain

$$\operatorname{Re}\left[i \frac{d}{ds}\left(\frac{\partial w_1}{\partial x} - i\frac{\partial w_1}{\partial y}\right)\right] = \operatorname{Re}\left[i \frac{d}{ds}\left(\frac{\partial w_2}{\partial x} - i\frac{\partial w_2}{\partial y}\right)\right]. \tag{4.134}$$

Adding the two equations (4.132) for $k = 1$ and $k = 2$, and taking (4.134) into

account, we obtain

$$\frac{N}{E_1F} - \operatorname{Re}\left[i\frac{d}{ds}(u_1 + iv_1)\right] = -\frac{N}{E_1F} + \operatorname{Re}\left[i\frac{d}{ds}(u_2 + iv_2)\right]. \tag{4.135}$$

By virtue of (4.134) and (4.135), we can rewrite (4.132) and (4.133) as follows:

$$\left.\begin{aligned} -h_0 \operatorname{Re}\left[i\frac{d}{ds}\left(\frac{\partial w_1}{\partial x} - i\frac{\partial w_1}{\partial y}\right)\right] - \operatorname{Re}\left[i\frac{d}{ds}(u_2 + iv_2)\right] + \frac{N}{E_1F} = 0 \\ h_0 \operatorname{Re}\left[i\frac{d}{ds}\left(\frac{\partial w_2}{\partial x} - i\frac{\partial w_2}{\partial y}\right)\right] - \operatorname{Re}\left[i\frac{d}{ds}(u_1 + iv_1)\right] + \frac{N}{E_1F} = 0 \\ -\operatorname{Re}\left[i\frac{d}{ds}\left(\frac{\partial w_1}{\partial x} - i\frac{\partial w_1}{\partial y}\right)\right] = \frac{L_n}{A} \\ -\operatorname{Re}\left[i\frac{d}{ds}\left(\frac{\partial w_2}{\partial x} - i\frac{\partial w_2}{\partial y}\right)\right] = \frac{L_n}{A} \end{aligned}\right\}. \tag{4.136}$$

Let

$$I_k(s) = -\int_0^s \left(m_k - i\int_0^{s_1} p_k ds_2\right)\dot{\bar{t}}\, ds_1 \qquad (k = 0, 1, 2), \tag{4.137}$$

$$I(s) = -\int_0^s \left(m - i\int_0^{s_1} p ds_2\right)\dot{\bar{t}}\, ds_1. \tag{4.138}$$

Substituting (4.126) in (4.138) and using (4.137), we obtain

$$I(s) = I_2(s) - I_1(s) - h_0\int_0^s [(N_1 - iT_1) + (N_2 - iT_2)]\dot{\bar{t}}\, ds, \tag{4.139}$$

where

$$\begin{aligned} (N_1 - iT_1)\dot{\bar{t}} &= ih_1(X_n^{(1)} - iY_n^{(1)}), \\ (N_2 - iT_2)\dot{\bar{t}} &= -ih_2(X_n^{(2)} - iY_n^{(2)}), \end{aligned} \tag{4.140}$$

where $X_n^{(k)}$, $Y_n^{(k)}$ $(k = 1, 2)$ are, respectively, the projections of the stresses on the x- and y-axes.

Substituting (4.140) in (4.139), we find

$$I(s) = I_2(s) - I_1(s) - ih_0h_1\int_0^s (X_n^{(1)} - iY_n^{(1)})\, ds + ih_0h_2\int_0^s (X_n^{(2)} - iY_n^{(2)})\, ds. \tag{4.141}$$

We shall now express the displacement components of plates I and II by formulas similar to (1.76) and (4.19):

$$\frac{1}{2}\left(\frac{\partial w_k}{\partial x} - i\frac{\partial w_k}{\partial y}\right) = \overline{\varphi_k^*(t)} + \bar{t}\varphi_k^{*\prime}(t) + \psi_k^*(t) + \frac{1}{2}\left(\frac{\partial w^0}{\partial x} - i\frac{\partial w^0}{\partial y}\right),$$

$$2\mu_k(u_k + iv_k) = \varkappa_1^{(k)}\overline{\varphi_k(t)} - \bar{t}\varphi_k'(t) - \psi_k(t) \qquad (k = 1, 2), \tag{4.142}$$

where w^0 is the known basic deflection of the unembossed plate at the same load, $\varkappa_1^{(k)} = \dfrac{3 - \nu_k}{1 + \nu_k}$, ν_k and μ_k are the elastic constants of the k-th plate.

$$-2iA\dot{t}\,\mathrm{Re}\left\{\dot{t}\,\frac{d}{ds}\left[\overline{\varphi_2^*(t)}+\bar{t}\varphi_2^{*\prime}(t)+\psi_2^*(t)\right]\right\}=$$

$$=tA\dot{t}\,\mathrm{Re}\left\{\dot{t}\,\frac{d}{ds}\left(\frac{\partial w^0}{\partial x}-i\frac{\partial w^0}{\partial y}\right)\right\}+iC_1\dot{t}-C_2, \tag{4.148}$$

$$\sum_{k=1}^{2}h_k\left[\overline{\varphi_k(t)}+\bar{t}\varphi_k'(t)+\psi_k(t)\right]-$$

$$-\frac{tE_1F}{2\mu_1}\dot{t}\,\mathrm{Re}\left\{\dot{t}\,\frac{d}{ds}\left[\varkappa_1^{(1)}\overline{\varphi_1(t)}-\bar{t}\varphi_1'(t)-\psi_1(t)\right]\right\}+$$

$$+2iE_1Fh_0\dot{t}\,\mathrm{Re}\left\{\dot{t}\,\frac{d}{ds}\left[\overline{\varphi_2^*(t)}+\bar{t}\varphi_2^{*\prime}(t)+\psi_2^*(t)\right]\right\}=$$

$$=ih_0E_1F\dot{t}\,\mathrm{Re}\left[\dot{t}\,\frac{d}{ds}\left(\frac{\partial w^0}{\partial x}-i\frac{\partial w^0}{\partial y}\right)\right]+C. \tag{4.149}$$

The boundary conditions (4.146) to (4.149) are generalizations of (4.23), (4.43), and (4.44). If we set in (4.146) to (4.149): $h_0=0$ and $h_1=h_2$, $D_1=D_2$, $\mu_1=\mu_2$ (i.e., we assume that the rib axis lies in the common middle plane of plates I and II, whose thicknesses and elastic constants are equal), we obtain (4.43) and (4.44). The terms behind $\frac{d}{ds}$ in (4.147) and (4.149) have opposite signs, since the directions of the coordinate axes in Figures 46 and 45 differ (this means, in fact, that Γ is described in opposite senses).

For $h_0=0$, $h_1=\mu_1=D_1=0$, we obtain (4.23) for the case of a rib reinforcing the plate edge. If $h_1=\mu_1=D_1=0$, but $h_0\neq 0$, we obtain the boundary conditions for the case when the edge of the hole is reinforced by a rib whose axis does not lie in the middle plane of the plate, while one of the principal axes of inertia of its cross section is parallel to the middle plane (rib on one side, asymmetrical relative to the middle plane).

If the functions $\varphi_k^*(z)$, $\psi_k^*(z)$ have been found, we can determine the deflections $w_k(x, y)$ from (4.10), except for constants defining the rigid-body displacements of the plate. These constants are determined from the condition

$$w_1(x, y)=w_2(x, y) \text{ on } \Gamma. \tag{4.150}$$

In the following section we shall consider as examples the axisymmetrical bending of a circular plate with central embossing. Boundary conditions (4.146) to (4.149) are exact for axisymmetrical loads, since in this case the twisting and bending moments in the plane of the rib vanish identically.

§ 8. AXISYMMETRICAL BENDING OF CIRCULAR AND ANNULAR PLATES WITH CONCENTRIC EMBOSSINGS AND ANNULAR RIBS. INFLUENCE FUNCTIONS

Structural elements such as circular and annular plates of uniform thickness or with stepped profiles are widely employed in mechanical and structural engineering. Such plates, whose rigidity and strength are increased by concentric embossings and thin ribs, are used for reservoir

We have, in addition, (4.19) and (1.65), according to which

$$i\int_0^s (X_n^{(k)} - iY_n^{(k)})\,ds = \overline{\varphi_k(t)} + \bar{t}\varphi_k'(t) + \psi_k(t) - C_{3k}, \tag{4.143}$$

$$I_k(s) = 2[\varkappa^{(k)}\overline{\varphi_k^*(t)} - \bar{t}\varphi_k^{*\prime}(t) - \psi_k^*(t)](1-\nu_k)D_k + I_0(s) + iC_k'\bar{t} - C_k'', \tag{4.144}$$

where $I_0(s)$ is obtained from (4.137) by substituting the bending moments m_0 and shearing forces p_0 corresponding to the basic deflection w^0; C' and C'' are, respectively, real and complex constants; $\varkappa^{(k)} = (3+\nu_k)/(1-\nu_k)$.

By analogy with (4.3) and (4.15), we obtain the equilibrium condition for part O_1s of the rib:

$$L_n = i\dot{t}[I(s) + iC_*\bar{t} - C_{**}],$$

$$N = -\dot{t}\left[h_1\int_0^s (X_n^{(1)} - iY_n^{(1)})\,ds + h_2\int_0^s (X_n^{(2)} - iY_n^{(2)})\,ds + C_0\right] = \tag{4.145}$$

$$= -\dot{t}\left[\sum_{k=1}^{2} h_k\int_0^s (X_n^{(k)} - iY_n^{(k)})\,ds + C_0\right].$$

Substituting (4.145) in (4.136), and taking (4.141) to (4.144) into account, we obtain finally four conditions /150/ on Γ for the four pairs of functions (φ_k, ψ_k), (φ_k^*, ψ_k^*) $(k = 1, 2)$, holomorphic in the corresponding domains of plates I and II (or for the four functions $\varphi(z)$, $\psi(z)$, $\varphi^*(z)$, $\psi^*(z)$, piecewise holomorphic in the domain of the entire plate with Γ as discontinuity line). These conditions are:

$$2\sum_{k=1}^{2}(-1)^k[\varkappa^{(k)}\overline{\varphi_k^*(t)} - \bar{t}\varphi_k^{*\prime}(t) - \psi_k^*(t)](1-\nu_k)D_k -$$

$$-h_0\sum_{k=1}^{2}(-1)^k h_k[\overline{\varphi_k(t)} + \bar{t}\varphi_k'(t) + \psi_k(t)] -$$

$$-2iA\dot{t}\,\mathrm{Re}\left\{\dot{t}\frac{d}{ds}[\overline{\varphi_1^*(t)} + \bar{t}\varphi_1^{*\prime}(t) + \psi_1^*(t)]\right\} =$$

$$= Ai\dot{t}\,\mathrm{Re}\left[\dot{t}\frac{d}{ds}\left(\frac{\partial w^0}{\partial x} - i\frac{\partial w^0}{\partial y}\right)\right] + iC_1\bar{t} - C_2, \tag{4.146}$$

$$\sum_{k=1}^{2} h_k[\overline{\varphi_k(t)} + \bar{t}\varphi_k'(t) + \psi_k(t)] -$$

$$-\frac{iE_1F}{2\mu_2}\dot{t}\,\mathrm{Re}\left\{\dot{t}\frac{d}{ds}[\varkappa_1^{(2)}\overline{\varphi_2(t)} - \bar{t}\varphi_2'(t) - \psi_2(t)]\right\} -$$

$$-2i\dot{t}E_1Fh_0\,\mathrm{Re}\left\{\dot{t}\frac{d}{ds}\left[\overline{\varphi_1^*(t)} + \bar{t}\varphi_1^{*\prime}(t) + \psi_1^*(t)\right]\right\} =$$

$$= ih_0E_1F\dot{t}\,\mathrm{Re}\left[\dot{t}\frac{d}{ds}\left(\frac{\partial w^0}{\partial x} - i\frac{\partial w^0}{\partial y}\right)\right] + C, \tag{4.147}$$

$$2\sum_{k=1}^{2}(-1)^k[\varkappa^{(k)}\overline{\varphi_k^*(t)} - \bar{t}\varphi_k^{*\prime}(t) - \psi_k^*(t)](1-\nu_k)D_k -$$

$$-h_0\sum_{k=1}^{2}(-1)^k h_k[\overline{\varphi_k(t)} + \bar{t}\varphi_k'(t) + \psi_k(t)] -$$

for the annular part (i.e., for $R \leqslant r \leqslant R_1$)

$$w_2 = w_2^0 - C_1 \ln \frac{\varrho}{r} - C_4 \left(\frac{\varrho^2}{R^2} - \frac{r^2}{R^2} \right), \tag{4.153}$$

$$v_{r_2} = B_1 \frac{R}{r} + B_2 \frac{r}{R}. \tag{4.154}$$

where w_i^0 $(i = 1, 2)$ are the basic deflections, under the same axisymmetrical load, of a thin plate of stepped profile without embossing, similarly supported along $r = \varrho$. These basic deflections are assumed to be known; we have to determine the additional states of stress and strain, i.e., the additional deflections, displacements v_{r_i}, and stresses of the plate, due to the embossing and the outer stiffening rib.

Expressions (4.151) and (4.153) satisfy automatically the conditions of support and of continuity of the deflections across the embossing web:

$$w_2 = 0 \text{ at } r = \varrho, \qquad w_1 = w_2 \text{ at } r = R,$$

since, by definition, the basic deflection w_2^0 satisfies similar conditions.

The six unknown coefficients A_1, B_1, B_2, C_1, C_2, C_4 defining the additional stresses in the plate will be found from the compatibility conditions for its different parts. The interactions of these parts, due to the additional stresses, are shown schematically in Figure 52.

Let γ_1, γ_2 and u_1, u_2 be, respectively, the angles of rotation of the radial sections and the radial displacements of points on the axes of the embossing web and the stiffening rib. The compatibility conditions (4.127) and (4.128) for the embossing web $r = R$ are in this case

$$\frac{dw_1}{dr} = \frac{dw_2}{dr} = \gamma_1, \tag{4.155}$$

$$v_{r_1} = u_1 + h_0 \gamma_1, \qquad v_{r_2} = u_1 - h_0 \gamma_1. \tag{4.156}$$

Similarly, for $r = R_1$ we have

$$\frac{dw_2}{dr} = \gamma_2, \tag{4.157}$$

$$v_{r_2} = u_2 - h_* \gamma_2. \tag{4.158}$$

Conditions (4.156) and (4.158) apply under the assumption that radial displacements of points on the axes of web and rib can be considered as displacements of points of the plate, located at distances h_0 and h_*, respectively, from its middle surface (cf. Figure 51).

Equation (4.158), and thus also the solution obtained below, are valid even when the centroids of section of the outer rib lie below the middle plane of the annular part of the plate (h_* is in this case negative).

bottoms, pipe flanges, cylinder covers, diaphragms, pistons, reinforced-concrete plates, etc. (Figures 51 and 52). It is of practical interest to study the influence of embossings on the rigidity and strength of such plates.

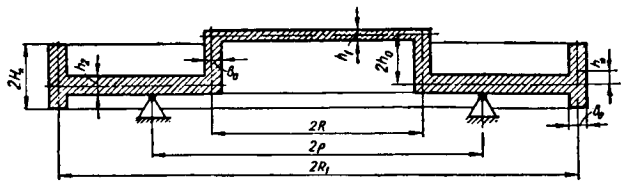


FIGURE 51

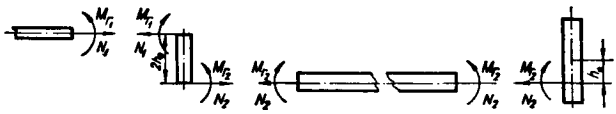


FIGURE 52.

We shall solve in this section the problem of the elastic equilibrium of a circular plate having a stepped profile with central embossing and annular stiffening rib, subjected to an arbitrary transverse axisymmetrical load.* Particular solutions will be derived for a whole class of engineering problems concerning circular and annular plates with one or two elastic annular ribs.

Despite the generalized formulation of the problem, the final results are presented in a compact form which is convenient for practical applications.

Consider an elastic isotropic circular plate with embossing, whose diametral section is shown in Figure 51. The plate is simply supported on the smooth contour $r = \varrho$ and subjected to an arbitrary transverse axisymmetrical load (r is the distance measured from the center of the plate). The plate is reinforced along the outer edge $r = R_1$ by a thin stiffening rib whose centroids of section lie at a distance h_* from the middle plane. We consider this embossed plate as a compound elastic body of revolution, consisting of a circular plate ($r \leqslant R$) of thickness h_1, an annular plate ($R \leqslant r \leqslant R_1$) of thickness h_2, and two thin circular elastic ribs [webs] of radii R and R_1. The distance between the middle planes of the plates (depth of embossing) is $2h_0$. The deflections w_i and the radial displacements v_{ri} of points of the middle plane of each plate are represented as follows /142/:

for the central part (i.e., for $0 \leqslant r \leqslant R$)

$$w_1 = w_1^0 + C_2\left(\frac{r^2}{R^2} - 1\right) - C_1 \ln\frac{\varrho}{R} - C_4\left(\frac{\varrho^2}{R^2} - 1\right), \qquad (4.151)$$

$$v_{r_1} = A_1\frac{r}{R}; \qquad (4.152)$$

* The results of this section were obtained by N.P. Fleishman and S.A. Grach /165-167/.

Substituting (4.151) to (4.154) in (4.155) to (4.158), we express all the six unknown coefficients through the four basic unknowns γ_1, γ_2, u_1, u_2:

$$\left.\begin{aligned} B_1 &= \frac{\eta}{\eta^2-1}[\eta(u_1-h_0\gamma_1)-(u_2-h_*\gamma_2)] \\ B_2 &= \frac{1}{\eta^2-1}[\eta(u_2-h_*\gamma_2)-(u_1-h_0\gamma_1)] \\ A_1 &= u_1+h_0\gamma_1, \qquad C_2=\frac{R}{2}(\gamma_1-\gamma_1^0) \\ C_1 &= \frac{R_1}{\eta^2-1}[\eta(\gamma_1-\gamma_1^0)-(\gamma_2-\gamma_2^0)] \\ C_4 &= \frac{R}{2(\eta^2-1)}[\eta(\gamma_2-\gamma_2^0)-(\gamma_1-\gamma_1^0)] \end{aligned}\right\}, \tag{4.159}$$

where

$$\eta=\frac{R_1}{R}, \qquad \gamma_1^0=\left(\frac{dw_1^0}{dr}\right)_{r=R}, \qquad \gamma_2^0=\left(\frac{dw_2^0}{dr}\right)_{r=R_1}. \tag{4.160}$$

Substituting (4.159) in (4.151) and (4.153), we obtain:

$$\left.\begin{aligned} w_1 &= w_1^0+\frac{R_1}{2(\eta^2-1)}\left\{(\gamma_2-\gamma_2^0)\left(2\ln\frac{\varrho}{R}+1-\frac{\varrho^2}{R^2}\right)+\right. \\ &\left.+(\gamma_1-\gamma_1^0)\eta\left[(\eta^2-1)\frac{r^2}{R_1^2}-\left(2\ln\frac{\varrho}{R}+1-\frac{\varrho^2}{R_1^2}\right)\right]\right\} \\ w_2 &= w_2^0+\frac{R_1}{2(\eta^2-1)}\left[\eta(\gamma_1-\gamma_1^0)\left(2\ln\frac{r}{\varrho}+\frac{\varrho^2-r^2}{R_1^2}\right)-\right. \\ &\left.-(\gamma_2-\gamma_2^0)\left(2\ln\frac{r}{\varrho}+\frac{\varrho^2-r^2}{R^2}\right)\right] \end{aligned}\right\}. \tag{4.161}$$

We can similarly express through γ_1, γ_2, u_1, u_2 the radial and tangential bending moments M_{ri}, $M_{\theta i}$ acting on the central (subscript "1") and annular (subscript "2") parts of the plate. According to (1.19), we have

$$\left.\begin{aligned} M_{ri} &= -D_i\left(\frac{d^2w_i}{dr^2}+\frac{\nu_i}{r}\cdot\frac{dw_i}{dr}\right) \\ M_{\theta i} &= -D_i\left(\frac{1}{r}\cdot\frac{dw_i}{dr}+\nu_i\frac{d^2w_i}{dr^2}\right) \end{aligned}\right\} \quad (i=1,2), \tag{4.162}$$

where $D_i=E_ih_i^3/12(1-\nu_i^2)$ is the flexural rigidity of the corresponding part of the plate; E_i is the modulus of elasticity; ν_i is Poisson's ratio.

Substitution of (4.161) in (4.162) yields

$$\left.\begin{aligned} M_{r1} &= M_{r1}^0-\frac{D_1}{R}(1+\nu_1)(\gamma_1-\gamma_1^0) \\ M_{\theta 1} &= M_{\theta 1}^0-\frac{D_1}{R}(1+\nu_1)(\gamma_1-\gamma_1^0) \\ M_{r2} &= M_{r2}^0+\frac{D_2}{R_1(\eta^2-1)}\left\{\eta(\gamma_1-\gamma_1^0)\left[(1+\nu_2)+\right.\right. \end{aligned}\right\} \tag{4.163}$$

$$+(1-\nu_2)\frac{R_1^2}{r^2}\Big]-(\gamma_2-\gamma_2^0)\Big[\eta^2(1+\nu_2)+(1-\nu_2)\frac{R_1^2}{r^2}\Big]\Big\}\Big\}$$

$$M_{\theta 2}=M_{\theta 2}^0-\frac{D_2}{R_1(\eta^2-1)}\Big\{\eta(\gamma_1-\gamma_1^0)\Big[(1-\nu_2)\frac{R_1^2}{r^2}-$$

$$-(1+\nu_2)\Big]+(\gamma_2-\gamma_2^0)\Big[\eta^2(1+\nu_2)-(1-\nu_2)\frac{R_1^2}{r^2}\Big]\Big\}\Big\} \qquad (4.163)$$
cont'd

where M_{ri}^0 and $M_{\theta i}^0$ are the known bending moments in the stepped plate without embossment or rib, obtained from (4.162) when w_i is replaced by w_i^0.

In addition to the bending stresses, the plate is also in a state of plane stress. The radial and tangential forces N_i and T_i, which depend on the displacements v_{ri} ((4.152) and (4.154\)) are according to the two-dimensional theory of elasticity:

$$\left.\begin{aligned} N_1 &= T_1=\frac{E_1h_1}{R(1-\nu_1)}A_1 \\ N_2 &= \frac{E_2h_2}{R(1-\nu_2^2)}\Big[B_2(1+\nu_2)-B_1(1-\nu_2)\frac{R^2}{r^2}\Big] \\ T_2 &= \frac{E_2h_2}{R(1-\nu_2^2)}\Big[B_2(1+\nu_2)+B_1(1-\nu_2)\frac{R^2}{r^2}\Big] \end{aligned}\right\}. \qquad (4.164)$$

Substituting (4.159) in (4.164), we obtain

$$\left.\begin{aligned} N_1 &= T_1=\frac{E_1h_1}{R(1-\nu_1)}(u_1+h_0\gamma_1) \\ N_2 &= \frac{E_2h_2}{R(1-\nu_2^2)(\eta^2-1)}\Big\{\eta(u_2-h_*\gamma_2)\Big[(1+\nu_2)+ \\ &+(1-\nu_2)\frac{R^2}{r^2}\Big]-(u_1-h_0\gamma_1)\Big[(1+\nu_2)+(1-\nu_2)\frac{R_1^2}{r^2}\Big]\Big\} \\ T_2 &= \frac{E_2h_2}{R(1-\nu_2^2)(\eta^2-1)}\Big\{\eta(u_2-h_*\gamma_2)\Big[(1+\nu_2)- \\ &-(1-\nu_2)\frac{R^2}{r^2}\Big]-(u_1-h_0\gamma_1)\Big[(1+\nu_2)-(1-\nu_2)\frac{R_1^2}{r^2}\Big]\Big\} \end{aligned}\right\}. \qquad (4.165)$$

For axisymmetrical strains of embossing and stiffening rib (Figure 52), we have

$$\gamma_1=\frac{m^{(1)}R^2}{A},\quad \gamma_2=\frac{m^{(2)}R_1^2}{A_*},\quad u_1=\frac{N^{(1)}R^2}{EF},\quad u_2=\frac{N^{(2)}R_1^2}{E_*F_*}. \qquad (4.166)$$

where $m^{(i)}$ and $N^{(i)}$ are, respectively, the bending moments and radial forces per unit length of rib axis ($i=1,2$); $A=EI$ is the flexural, and EF the tensile rigidity of the embossing web; $A_*=E_*I_*$ is the flexural, and E_*F_* the tensile rigidity of the outer rib; I and I_* are the moments of inertia of the web and rib sections relative to their central axes, parallel to the middle plane of the plate; E and E_* are the moduli of elasticity for web and rib, F and F_* are the corresponding cross-sectional areas.

Formulas (4.166) are valid for a thin rib whose cross section is not very high. The limits of applicability of the first two expressions (4.166) have been investigated in detail in /131/.

In our case we have, according to (4.125) and (4.126), (cf. Figure 52)

$$\left.\begin{aligned} m^{(1)} &= [(M_{r1} - M_{r2}) - h_0 (N_1 + N_2)]_{r=R} \\ m^{(2)} &= (M_{r2} + h_* N_2)_{r=R_1} \\ N^{(1)} &= (N_2 - N_1)_{r=R}, \quad N^{(2)} = -(N_2)_{r=R_1} \end{aligned}\right\}. \tag{4.167}$$

Substituting (4.167) in the right-hand sides of (4.166), and taking (4.163) and (4.165) into account, we obtain the following system of four equations:

$$\alpha_{j1} h_0 \gamma_1 + \alpha_{j3} h_* \gamma_2 + \alpha_{j2} u_1 + \alpha_{j4} u_2 = \alpha_{j0} h_0 \gamma_1^0 + \alpha_{j*} h_* \gamma_2^0 \quad (j = 1, 2, 3, 4), \tag{4.168}$$

where

$$\left.\begin{aligned} \alpha_{11} &= \eta(\eta^2 - 1)\delta + [(1 + \nu_2) + \eta^2(1 - \nu_2)]\left(1 + 12\frac{h_0^2}{h_2^2}\right) + \\ &+ \lambda(1 + \nu_1)(\eta^2 - 1)\left(1 + 12\frac{h_0^2}{h_1^2}\right) \\ \alpha_{12} &= 12\left\{\lambda(1 + \nu_1)(\eta^2 - 1)\frac{h_0^2}{h_1^2} - \frac{h_0^2}{h_2^2}[(1 + \nu_2) + \eta^2(1 - \nu_2)]\right\}, \\ \alpha_{13} &= -2\eta\left(\frac{h_0}{h_*} + 12\frac{h_0^2}{h_2^2}\right), \quad \alpha_{14} = 24\eta\frac{h_0^2}{h_2^2} \\ \alpha_{21} &= \beta_2(1 + \nu_2)(\eta^2 - 1) - [(1 + \nu_2) + \eta^2(1 - \nu_2)] \end{aligned}\right\}, \tag{4.169}$$

$$\left.\begin{aligned} \alpha_{22} &= (\eta^2 + 1) + (\eta^2 - 1)[\beta_1\eta - \nu_2 + (1 + \nu_2)\beta_2] \\ \alpha_{23} &= -\alpha_{24} = \alpha_{31} = -\alpha_{32} = 2\eta \\ \alpha_{33} &= -[\eta^2(1 + \nu_2) + (1 - \nu_2)] \\ \alpha_{34} &= (\eta^2 + 1) + (\eta^2 - 1)(\beta_2 + \nu_2) \\ \alpha_{41} &= -2\eta\left(\frac{h_*}{h_0} + 12\frac{h_*^2}{h_2^2}\right), \quad \alpha_{42} = 24\eta\frac{h_*^2}{h_2^2} \\ \alpha_{43} &= (\eta^2 - 1)\delta_* + [\eta^2(1 + \nu_2) + (1 - \nu_2)]\left(1 + 12\frac{h_*^2}{h_2^2}\right) \\ \alpha_{44} &= -12\frac{h_*^2}{h_2^2}[\eta^2(1 + \nu_2) + (1 - \nu_2)] \\ \alpha_{10} &= \eta^2(1 - \nu_2) + (1 + \nu_2) + \lambda(1 + \nu_1)(\eta^2 - 1) \\ \alpha_{1*} &= -2\eta\frac{h_0}{h_*}, \quad \alpha_{2*} = \alpha_{3*} = \alpha_{20} = \alpha_{30} = 0 \\ \alpha_{4*} &= \eta^2(1 + \nu_2) + (1 - \nu_2), \quad \alpha_{40} = -2\eta\frac{h_*}{h_0}, \quad \lambda = \frac{D_1}{D_2} \\ \beta_1 &= \frac{EF(1 - \nu_2^2)}{E_2 h_2 R_1}, \quad \beta_2 = \frac{E_* F_*(1 - \nu_2^2)}{E_2 h_2 R_1}, \\ \beta_3 &= \frac{E_1 h_1(1 - \nu_2)}{E_2 h_2(1 - \nu_1)} = \lambda\frac{h_2^2(1 + \nu_1)}{h_1^2(1 + \nu_2)} \end{aligned}\right\}, \tag{4.169a}$$

$$\delta = \frac{A}{R_1 D_2}, \quad \delta_* = \frac{A_*}{R_1 D_2}. \tag{4.170}$$

Solution of system (4.168) yields:

$$\left.\begin{aligned}
\gamma_1 &= \frac{1}{\Delta}\left[(\alpha_{10}\omega_1 + \alpha_{40}\omega_2)\gamma_1^0 + \frac{1}{h_0}(h_*\alpha_{4_*}\omega_2 - 2\eta h_0\omega_1)\gamma_2^0\right] \\
\gamma_2 &= \frac{1}{h_*\omega_1}[h_0\omega_4\gamma_1 + \omega_6(h_0\alpha_{40}\gamma_1^0 + h_*\alpha_{4_*}\gamma_2^0)] \\
u_1 &= \frac{1}{\omega_6}(h_0\omega_7\gamma_1 + h_*\omega_8\gamma_2) \\
u_2 &= \frac{1}{\alpha_{23}\omega_6}(h_0\omega_9\gamma_1 + h_*\omega_{10}\gamma_2)
\end{aligned}\right\}, \qquad (4.171)$$

where

$$\left.\begin{aligned}
\Delta &= \alpha_{11}\omega_1 + \alpha_{12}\omega_3 + \alpha_{13}\omega_4 + \alpha_{14}\omega_5 \\
\omega_1 &= \alpha_{24}[\alpha_{42}(\alpha_{33}+\alpha_{34}) + \alpha_{23}(\alpha_{43}+\alpha_{44})] + \alpha_{22}(\alpha_{34}\alpha_{43} - \alpha_{33}\alpha_{44}) \\
\omega_2 &= \alpha_{24}[\alpha_{32}(\alpha_{13}+\alpha_{14}) - \alpha_{12}(\alpha_{33}+\alpha_{34})] + \alpha_{22}(\alpha_{14}\alpha_{33} - \alpha_{13}\alpha_{34}) \\
\omega_3 &= \alpha_{24}[\alpha_{31}(\alpha_{43}+\alpha_{44}) - \alpha_{41}(\alpha_{33}+\alpha_{34})] + \alpha_{21}(\alpha_{33}\alpha_{44} - \alpha_{34}\alpha_{43}) \\
\omega_4 &= \alpha_{24}[\alpha_{32}(\alpha_{41}+\alpha_{42}) - \alpha_{44}(\alpha_{21}+\alpha_{22})] + \alpha_{34}(\alpha_{21}\alpha_{42} - \alpha_{22}\alpha_{41}) \\
\omega_5 &= \alpha_{24}[\alpha_{32}(\alpha_{41}+\alpha_{42}) + \alpha_{43}(\alpha_{21}+\alpha_{22})] + \alpha_{33}(\alpha_{22}\alpha_{41} - \alpha_{21}\alpha_{42}) \\
\omega_6 &= \alpha_{22}\alpha_{34} - \alpha_{24}\alpha_{32}, \qquad \omega_7 = \alpha_{24}\alpha_{31} - \alpha_{21}\alpha_{34} \\
\omega_8 &= \alpha_{24}(\alpha_{33}+\alpha_{34}), \qquad \omega_9 = \alpha_{23}\alpha_{24}(\alpha_{21}+\alpha_{22}) \\
\omega_{10} &= \alpha_{24}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{22}\alpha_{34}(\alpha_{23} - \alpha_{22})
\end{aligned}\right\}. \qquad (4.172)$$

Substituting (4.171) in (4.161), we obtain the deflections:

$$w_i = w_i^0 + w_i^{(1)}\gamma_1^0 + w_i^{(2)}\gamma_2^0 \qquad (i = 1, 2), \qquad (4.173)$$

where $w_i^{(1)}$ and $w_i^{(2)}$ are influence functions which depend only on the geometrical and elastic parameters of plate and rib, but not on the load. Hence, for a given embossed plate, these influence functions need be calculated only once and tabulated for all cases of axisymmetrical loading. In (4.173) only the basic deflections w_i^0 and angles γ_1^0 and γ_2^0 depend on the load; these are known if we have solved the problem for a circular plate of stepped profile. Expressions for the influence functions are easily obtained, but will not be derived here for the general case, since they are rather involved.

The bending stresses at the surfaces of the plate are obtained from (4.163) in the form

$$\sigma_{ri} = \pm\frac{6M_{ri}}{h_i^2}, \qquad \sigma_{\theta i} = \pm\frac{6M_{\theta i}}{h_i^2}. \qquad (4.174)$$

The normal stresses σ_{ri}^* and $\sigma_{\theta i}^*$, uniformly distributed over the plate thickness, are obtained from (4.164) in the form:

$$\sigma_{ri}^* = \frac{N_i}{h_i}, \qquad \sigma_{\theta i}^* = \frac{T_i}{h_i}. \qquad (4.175)$$

The total normal stresses in the plate, obtained by superposing the bending stresses (4.174) and normal stresses (4.175), are

$$\left.\begin{aligned}
\sigma_{ri}^{\text{tot}} &= \sigma_{ri}^0 + \sigma_{ri}^{(1)}\gamma_1^0 + \sigma_{ri}^{(2)}\gamma_2^0 \\
\sigma_{\theta i}^{\text{tot}} &= \sigma_{\theta i}^0 + \sigma_{\theta i}^{(1)}\gamma_1^0 + \sigma_{\theta i}^{(2)}\gamma_2^0
\end{aligned}\right\}, \qquad (4.176)$$

where σ_{rl}^0 and $\sigma_{\theta l}^0$ are the known normal stresses corresponding to the basic deflections w_l^0.

The influence functions

$$\sigma_{rl}^{(1)},\ \sigma_{rl}^{(2)},\ \sigma_{\theta l}^{(1)},\ \sigma_{\theta l}^{(2)} \qquad (l = 1, 2)$$

do not depend on the load, but only on the elastic properties of the structure (plate, embossing rib) and its dimensions.

The above solution is applicable without any modification to the built-up plate whose diametral section is shown in Figure 53,a, at any transverse axisymmetrical load.

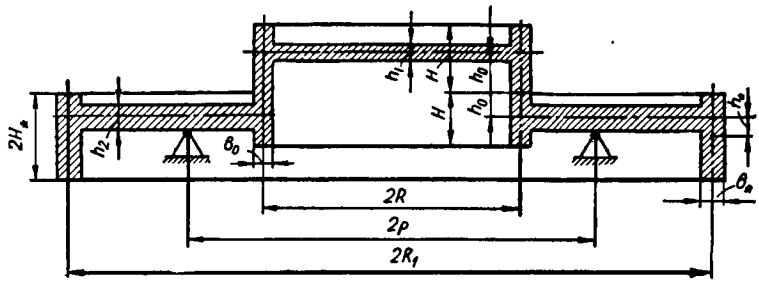


FIGURE 53.a.

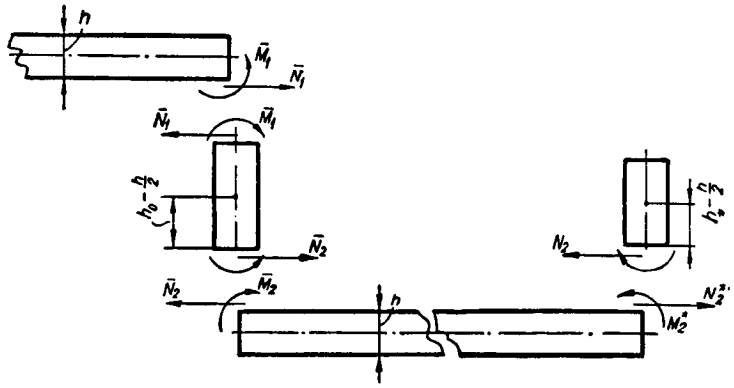


FIGURE 53,b.

If we assume that the plate is joined to the thin rib along the line of intersection between the middle cylinder of the rib and the plate surface adjoining it (Figure 53,b), as recommended by S. N. Sokolov /131/, (4.167) remains valid for $h_1 = h_2 = h$. In fact, in this case (Figure 53,b)

$$\left.\begin{aligned} M_{r1}|_{r=R} &= \overline{M}_1 + \frac{h}{2}\overline{N}_1, \quad M_{r2}|_{r=R} = \overline{M}_2 - \frac{h}{2}\overline{N}_2 \\ M_{r2}|_{r=R_1} &= M_2^* - \frac{h}{2}N_2^* \\ N_1|_{r=R} &= \overline{N}_1, \quad N_2|_{r=R} = \overline{N}_2, \quad N_2|_{r=R_1} = N_2^* \end{aligned}\right\}, \qquad (4.176a)$$

$$m_1=(\bar{M}_1-\bar{M}_2)-\left(h_0-\frac{h}{2}\right)(\bar{N}_1+\bar{N}_2),\quad n_1=\bar{N}_2-\bar{N}_1$$
$$m_2=M_2^*+\left(h_*-\frac{h}{2}\right)N_2^*,\qquad n_2=-N_2^*\Bigg\};\tag{4.176b}$$

substituting from (4.176 a) for $\bar{M}_1, \bar{M}_2, M_2^*, \bar{N}_1, \bar{N}_2$ and N_2^* in (4.176b), we obtain (4.167).

Since in the case considered the compatibility conditions (4.155) to (4.158) and equations (4.166) remain unaltered, we conclude that (4.171) remains valid also for the system shown in Figure 53,b.

We shall consider some particular cases.

Circular plate with central embossing, reinforced by a symmetrical outer rib

Let one of the principal axes of inertia of the cross section of the outer rib lie in the middle plane of the annular part of the plate, i.e., $h_* = 0$. (Figure 54). We obtain in this case from (4.171)

$$\left.\begin{aligned}
\gamma_1&=\frac{1}{\Delta_1}(\alpha_5\alpha_{10}-4\eta^2)\,\gamma_1^0-\frac{2\eta}{\Delta_1}(\alpha_{33}+\alpha_5)\gamma_2^0\\
\gamma_2&=\frac{1}{\alpha_5}[2\eta(\gamma_1-\gamma_1^0)-\alpha_{33}\gamma_2^0]\\
u_1&=\frac{\alpha_{21}\alpha_{34}+4\eta^2}{4\eta^2-\alpha_{22}\alpha_{34}}h_0\gamma_1\\
u_2&=\frac{2\eta(\alpha_{21}+\alpha_{22})}{4\eta^2-\alpha_{22}\alpha_{34}}h_0\gamma_1
\end{aligned}\right\},\tag{4.177}$$

where

$$\left.\begin{aligned}
&\alpha_5=\eta^2(1+\nu_2+\delta_*)+(1-\nu_2-\delta_*)\\
&\Delta_1=\frac{1}{\alpha_{22}\alpha_{34}-4\eta^2}[(\alpha_{22}\alpha_{34}-4\eta^2)(\alpha_{11}\alpha_5-4\eta^2)-\\
&-\alpha_{12}\alpha_5(4\eta^2+\alpha_{21}\alpha_{34})-2\eta\alpha_{14}\alpha_5(\alpha_{21}+\alpha_{22})]
\end{aligned}\right\}.\tag{4.178}$$

Embossed circular plate hinged along the circle $r=R_1$ (Figure 54,a).

In this case $u_2=0$, which is equivalent to $E_*F_*\to\infty$ in (4.166) or $\beta_2\to\infty$ in (4.169 a). In the limit, we obtain from (4.177)

$$\left.\begin{aligned}
\gamma_1&=\frac{1}{\Delta_2}[(\alpha_5\alpha_{10}-4\eta^2)\gamma_1^0-2\eta(\alpha_{33}+\alpha_5)\gamma_2^0]\\
\gamma_2&=\frac{1}{\alpha_5}[2\eta(\gamma_1-\gamma_1^0)-\alpha_{33}\gamma_2^0]\\
u_1&=-\frac{\alpha_{21}}{\alpha_{22}}h_0\gamma_1,\qquad u_2=0
\end{aligned}\right\},\tag{4.179}$$

where

$$\Delta_3 = \frac{1}{\alpha_{22}}[\alpha_{22}(\alpha_{11}\alpha_5 - 4\eta^2) - \alpha_{12}\alpha_5\alpha_{21}].$$

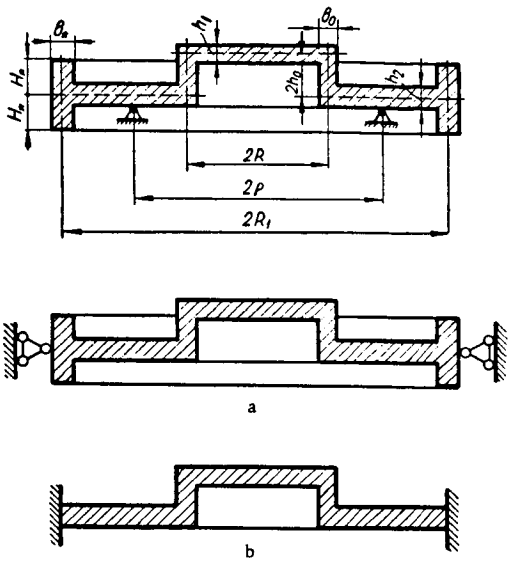


FIGURE 54.

If, in addition, the flexural rigidity of the plate and its material are uniform, i.e., $D_1 = D_2 = D$, $\nu_1 = \nu_2 = \nu$, and $h_1 = h_2 = h$, then, according to (4.169a), $\beta_3 = \lambda = 1$ and (4.179) becomes

$$\left.\begin{aligned} \gamma_1 &= \frac{2}{\Delta_3}\eta^2[2\eta + \beta_1(\eta^2 - 1)][\eta(1 + \nu + \delta_*)\gamma_1^0 - \delta_*\gamma_2^0] \\ \gamma_2 &= \frac{2\eta(\gamma_1 - \gamma_1^0) + \alpha_{4_*}\gamma_2^0}{\eta^3(1 + \nu + \delta_*) + (1 - \nu - \delta_*)} \\ u_1 &= \frac{2h_0(1 + \nu - \nu\eta^2)}{\eta(2\eta + \beta_1\eta^2 - \beta_1)}\gamma_1, \qquad u_2 = 0 \end{aligned}\right\}, \tag{4.180}$$

where

$$\Delta_3 = \eta^2[2\eta + \beta_1(\eta^2 - 1)][\alpha_5\delta + 2\eta(1 + \nu + \delta_*)] + 24\,\alpha_5\frac{h_0^3}{h^3}[\eta^2\beta_1 + 2(1 + \nu)^2 + 2\eta^2(1 - \nu^2)].$$

Homogeneous circular plate with embossing of the same thickness, rigidly clamped along the circle $r = R_1$ (Figure 54,b).

Letting $\delta_* \to \infty$ in (4.180), we obtain

$$\left.\begin{aligned} \gamma_1 &= \frac{2}{\Delta_{12}}\eta^2(2\eta+\beta_1\eta^2-\beta_1)(\eta\gamma_1^0-\gamma_2^0) \\ u_1 &= \frac{2h_0(1+\nu-\nu\eta^2)}{\eta(2\eta+\beta_1\eta^2-\beta_1)}\gamma_1 \\ \gamma_2 &= u_2 = 0 \end{aligned}\right\}, \tag{4.181}$$

where

$$\Delta_{12} = \eta^3[2\eta+\beta_1(\eta^2-1)][(\eta^2-1)\delta+2\eta] + \\ + 24\frac{h_0^2}{h^2}(\eta^2-1)[\eta^3\beta_1+2(1+\nu)^2+2\eta^2(1-\nu^2)].$$

Setting $h_0 = 0$ in (4.181), we obtain the solution corresponding to axisymmetrical bending of a circular plate with concentric rib, rigidly clamped at the outer edge. This solution was obtained by a different procedure in § 14 of Chapter II.

Circular embossed plate

No outer rib is provided in this case, i.e., $h_* = \delta_* = 0$. The circular plate is reinforced only by a central embossing, and is supported at a distance ϱ from its center (Figure 55,a). The solution for this problem is given by (4.177), where we set $\delta_* = 0$. This is equivalent to replacing α_6 by $-\alpha_{33}$. We thus obtain

$$\left.\begin{aligned} \gamma_1 &= \frac{1}{\Delta_4}(\alpha_{33}\alpha_{10}+4\eta^2)\gamma_1^0 \\ \gamma_2 &= \frac{2\eta}{\alpha_{33}}(\gamma_1^0-\gamma_1)+\gamma_2^0 \\ u_1 &= \frac{4\eta^2+\alpha_{21}\alpha_{34}}{4\eta^2-\alpha_{22}\alpha_{34}}h_0\gamma_1, \qquad u_2 = \frac{2\eta(\alpha_{21}+\alpha_{22})}{4\eta^2-\alpha_{22}\alpha_{34}}h_0\gamma_1 \end{aligned}\right\}, \tag{4.182}$$

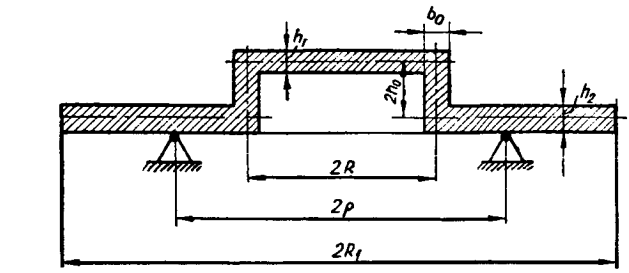
where

$$\Delta_4 = \frac{1}{4\eta^2-\alpha_{22}\alpha_{34}}[(\alpha_{11}\alpha_{33}+4\eta^2)(4\eta^2-\alpha_{22}\alpha_{34}) + \\ + \alpha_{12}\alpha_{23}(4\eta^2+\alpha_{21}\alpha_{34})+2\eta\alpha_{14}\alpha_{33}(\alpha_{21}+\alpha_{22})].$$

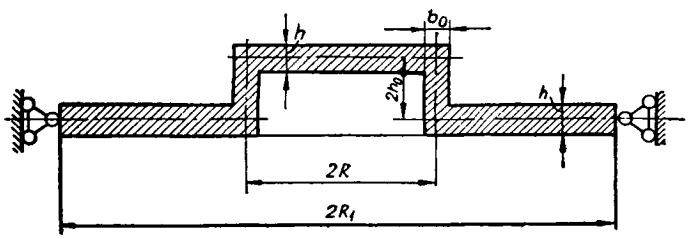
Since γ_2 and γ_1 are interrelated by (4.182), we obtain fairly simple expressions for the deflections (4.161), the bending stresses (4.174) at the plate surfaces, and the normal stresses:

$$\left.\begin{aligned} w_1 &= w_1^0+\bar{w}_1R_1\gamma_1^0 = w_1^0+\frac{R_1\eta(\gamma_1-\gamma_1^0)}{2[\eta^2(1+\nu_2)+(1-\nu_2)]}\times \\ &\times\left[(1-\nu_2)\frac{r^2-\varrho^2}{R_1^2}-(1+\nu_2)\left(1+2\ln\frac{\varrho}{R}-\frac{r^2}{R^2}\right)\right], \end{aligned}\right\} \tag{4.183}$$

$$
\left.
\begin{gathered}
w_2 = w_2^0 + \bar{w}_2 R_1 \gamma_1^0 = w_2^0 + \frac{R_1 \eta (\gamma_1 - \gamma_1^0)}{2[\eta^2(1+\nu_2)+(1-\nu_2)]} \times \\
\times \left[2(1+\nu_2) \ln \frac{r}{\varrho} + (1-\nu_2) \frac{r^2 - \varrho^2}{R_1^2} \right], \\
\pm \sigma_{r1} = \frac{6}{h_1^2} M_{r1}^0 + \bar{\sigma}_1 \frac{E_1 h_1}{R_1} \gamma_1^0 = \frac{6}{h_1^2} M_{r1}^0 - \frac{E_1 h_1}{2R(1-\nu_1)} (\gamma_1 - \gamma_1^0), \\
\pm \sigma_{\theta 1} = \frac{6}{h_1^2} M_{\theta 1}^0 + \bar{\sigma}_1 \frac{E_1 h_1}{R_1} \gamma_1^0 = \frac{6}{h_1^2} M_{\theta 1}^0 - \frac{E_1 h_1}{2R(1-\nu_1)} (\gamma_1 - \gamma_1^0), \\
\pm \sigma_{r2} = \frac{6}{h_2^2} M_{r2}^0 + \bar{\sigma}_{r2} \frac{E_2 h_2}{R_1} \gamma_1^0 = \frac{6}{h_2^2} M_{r2}^0 - \frac{E_2 h_2}{2R a_{33}} (\gamma_1 - \gamma_1^0) \left(\frac{R_1^2}{r^2} - 1 \right), \\
\pm \sigma_{\theta 2} = \frac{6}{h_2^2} M_{\theta 2}^0 + \bar{\sigma}_{\theta 2} \frac{E_2 h_2}{R_1} \gamma_1^0 = \frac{6}{h_2^2} M_{\theta 2}^0 + \frac{E_2 h_2}{2R a_{33}} (\gamma_1 - \gamma_1^0) \left(\frac{R_1^2}{r^2} + 1 \right), \\
\sigma_{r1}^* = \sigma_{\theta 1}^* = \bar{\sigma}_1^* \frac{E_1 h_0}{R_1} \gamma_1^0 (1 - \gamma_1^*) = \frac{E_1}{R(1-\nu_1)} (u_1 + h_0 \gamma_1), \\
\sigma_{r2}^* = \bar{\sigma}_{r2}^* \frac{E_2 h_0}{R_1} (1 - \gamma_1^*) \gamma_1^0 = \frac{E_2 [u_2 + \eta (h_0 \gamma_1 - u_1)]}{R_1 (1 - \nu_2^2)(\eta^2 - 1)} \times \\
\times \left[(1+\nu_2) + \frac{R_1^2}{r^2} (1-\nu_2) \right], \\
\sigma_{\theta 2}^* = \bar{\sigma}_{\theta 2}^* \frac{E_2 h_0}{R_1} (1 - \gamma_1^*) \gamma_1^0 = \frac{E_2 [u_2 + \eta (h_0 \gamma_1 - u_1)]}{R_1 (1 - \nu_2^2)(\eta^2 - 1)} \times \\
\times \left[(1+\nu_2) - \frac{R_1^2}{r^2} (1-\nu_2) \right], \\
\gamma_1^* = 1 - \frac{\gamma_1}{\gamma_1^0}.
\end{gathered}
\right\} \quad (4.183 \text{ cont'd})
$$



a



b

FIGURE 55.

The parameter γ_2^0 does not, of course, appear in these formulas.

Homogeneous circular plate with embossing of uniform thickness ($\nu_1 = \nu_2 = \nu$, $h_1 = h_2 = h$, $D_1 = D_2 = D$, $\beta_2 = \lambda = 1$) hinged along the outer edge $r = R_1$ (Figure 55,b). The solution is obtained by setting $u_2 = 0$ ($\beta_2 \to \infty$) in (4.182):

$$\left.\begin{aligned} \gamma_1 &= \frac{2}{\Delta_5}\eta^3(\eta^2-1)(1+\nu)[2\eta+\beta_1(\eta^2-1)]\gamma_1^0 \\ u_1 &= \frac{2h_0(1+\nu-\nu\eta^2)}{\eta(2\eta+\beta_1\eta^2-\beta_1)}\gamma_1, \quad u_2 = 0 \end{aligned}\right\}, \tag{4.184}$$

where

$$\Delta_5 = [2\eta^3+\beta_1\eta(\eta^2-1)](4\eta^2-\alpha_{11}\alpha_{33}) + 2\alpha_{12}\alpha_{33}(\nu\eta^3-1-\nu).$$

The coefficients α_{11} and α_{12} in (4.169) are here determined for $\lambda = 1$ and $h_1 = h_2 = h$. The deflections and stresses are again given by (4.183), with (4.184) being used.

Stepped plate with two annular ribs

A circular plate having a stepped profile is reinforced symmetrically to its middle plane by two thin ribs along the circles $r = R$ and $r = R_1$ (Figure 56). Setting $h_* = h_0 = 0$ in (4.171), we obtain finally

$$\left.\begin{aligned} \gamma_1 - \gamma_1^0 &= -\frac{1}{\Delta_6}\{[(\eta^2-1)\delta_* + \eta^2(1+\nu_2) + \\ &+ (1-\nu_2)]\eta\delta\gamma_1^0 + 2\eta\delta_*\gamma_2^0\} \\ \gamma_2 - \gamma_2^0 &= -\frac{1}{\Delta_6}\{2\eta^2\delta\gamma_1^0 + [\eta\delta(\eta^2-1) + (1+\nu_2) + \\ &+ \eta^2(1-\nu_2) + \lambda(1+\nu_1)(\eta^2-1)]\delta_*\gamma_2^0\} \end{aligned}\right\}, \tag{4.185}$$

where

$$\Delta_6 = [\eta\delta + \lambda(1+\nu_1)][(\eta^2-1)\delta_* + \eta^2(1+\nu_2) + (1-\nu_2)] + \\ + \delta_*[(1+\nu_2) + \eta^2(1-\nu_2)] + (1-\nu_2^2)(\eta^2-1).$$

As ought to be expected, the plate is in this case subjected only to bending stresses, while the normal stresses σ_{rl}^* and $\sigma_{\beta l}^*$ ((4.175)) and the forces T_l and N_l ((4.165)) in the middle surface, vanish identically.

For a homogeneous plate ($\nu_1 = \nu_2 = \nu$, $E_1 = E_2$) of uniform rigidity ($\beta_2 = \lambda = 1$), we obtain from (4.185)

$$\left.\begin{aligned} \gamma_1 - \gamma_1^0 &= \frac{1}{\Delta_7}\{[\eta^2(1+\nu+\delta_*) + (1-\nu-\delta_*)]\delta\gamma_1^0 + 2\delta_*\gamma_2^0\} \\ \gamma_2 - \gamma_2^0 &= \frac{1}{\Delta_7}\{2\eta\delta\gamma_1^0 + [\delta(\eta^2-1) + 2\eta]\delta_*\gamma_2^0\}, \quad u_1 = u_2 = 0 \end{aligned}\right\}, \tag{4.186}$$

where

$$\Delta_7 = -\eta(\eta\delta + 2)(1 + \nu + \delta_*) - \delta(1 - \nu - \delta_*).$$

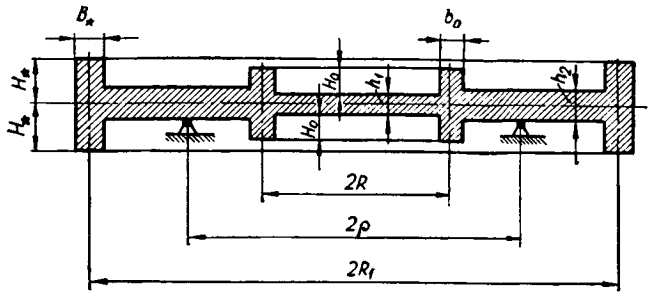


FIGURE 56.

A stepped plate is reinforced by a single inner or outer stiffening rib. The solution is obtained from (4.185) for $\delta_* = 0$ or $\delta = 0$, respectively (cf. § 12 of Chapter II).

Annular plate with reinforced edges

A thin annular plate, reinforced by ribs at the edges, is simply supported along the circle $r = \varrho$ (Figure 57,a) and loaded symmetrically with respect to its center. The solution for this case is obtained from (4.171), with

$$h_1 = \lambda = \beta_3 = 0. \tag{4.187}$$

The outer annular rib is symmetrical to the middle plane, and the plate is hinged along the edge $r = R_1$ on a fixed support (Figure 57,b). The solution is obtained from (4.179) by substituting (4.187):

$$\left.\begin{aligned}
\gamma_1 &= \frac{1}{\Delta_8}\{\alpha_5[\eta^2(1-\nu_2) + (1+\nu_2)] - 4\eta^2\}\gamma_1^0 - \\
&- \frac{2\eta}{\Delta_8}(\alpha_5 + \alpha_{33})\gamma_2^0 \\
\gamma_2 &= \frac{1}{\alpha_5}[2\eta(\gamma_1 - \gamma_1^0) - \alpha_{33}\gamma_2^0] \\
u_1 &= \frac{(1+\nu_2) + \eta^2(1-\nu_2)}{(1+\eta^2) + (\eta^2-1)(\beta_1\eta - \nu_2)} h_0\gamma_1 \\
u_2 &= 0
\end{aligned}\right\}, \tag{4.188}$$

where

$$\Delta_8 = \alpha_5\left[\eta(\eta^2-1)\delta + (1+\nu_2+\eta^2-\eta^2\nu_2)\left(1+12\frac{h_0^2}{h_2^2}\right)\right] -$$

$$-4\eta^2 - \frac{12\alpha_5[(1+\nu_2)+\eta^2(1-\nu_2)]^2}{(\eta^2+1)+(\eta^2-1)(\beta_1\eta-\nu_2)}\cdot\frac{h_0^2}{h_2^2}.$$

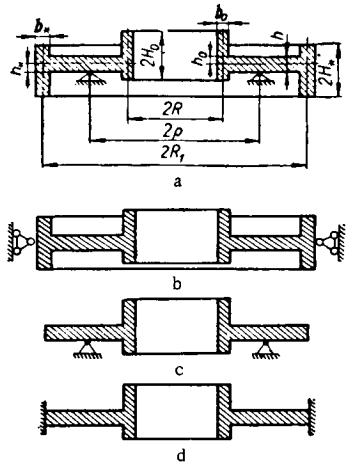


FIGURE 57.

No outer rib is provided (Figure 57,c), i.e., $\delta_* = 0$, and the plate is simply supported along the circle $r = \varrho$. Substituting (4.187) in (4.182), we obtain

$$\left.\begin{aligned}\gamma_1 &= \frac{1}{\Delta_9}[\alpha_{33}(\eta^2-\nu_2\eta^2+1+\nu_2)+4\eta^2]\gamma_1^0\\ u_1 &= \frac{4\eta^2-\alpha_{34}(1+\nu_2+\eta^2-\eta^2\nu_2)}{4\eta^2-\alpha_{34}[(\eta^2+1)+(\eta^2-1)(\beta_1\eta-\nu_2)]}h_0\gamma_1\\ u_2 &= \frac{2\eta^2\beta_1(\eta^2-1)h_0\gamma_1}{4\eta^2-\alpha_{34}[(\eta^2+1)+(\eta^2-1)(\beta_1\eta-\nu_2)]}\end{aligned}\right\}, \qquad (4.189)$$

where

$$\Delta_9 = 4\eta^2 + \alpha_{33}\left[\eta(\eta^2-1)\delta + (1+\nu_2+\eta^2-\eta^2\nu_2)\left(1+12\frac{h_0^2}{h_2^2}\right)\right] +$$

$$+\frac{1}{4\eta^2-\alpha_{34}[(\eta^2+1)+(\eta^2-1)(\beta_1\eta-\nu_2)]}\left\{2\eta^2\alpha_{14}\alpha_{23}\beta_1(\eta^2-1)-\right.$$

$$\left.-12\alpha_{33}\frac{h_0^2}{h_2^2}[(1+\nu_2)+\eta^2(1-\nu_2)][4\eta^2-\alpha_{34}(1+\nu_2+\eta^2-\eta^2\nu_2)]\right\}.$$

The deflections w_2 and the stresses σ_{r2}, $\sigma_{\theta 2}$, σ^*_{r2}, $\sigma^*_{\theta 2}$ are given by (4.183).

An annular plate is rigidly clamped at the outer edge and reinforced by a rib along the inner edge (Figure 57,d). Letting in (4.188) $\delta_* \to \infty$, we obtain

$$\left.\begin{aligned} \gamma_1 &= \frac{1}{\Delta_{10}}\{[\eta^2(1-\nu_2)+(1+\nu_2)]\gamma_1^0 - 2\eta\gamma_2^0\} \\ u_1 &= \frac{(1+\nu_2)+\eta^2(1-\nu_2)}{(1+\nu_2)+\eta^2(1-\nu_2)+\beta_1\eta(\eta^2-1)} h_0\gamma_1 \\ \gamma_2 &= u_2 = 0 \end{aligned}\right\}, \tag{4.190}$$

where

$$\Delta_{10} = \eta(\eta^2-1)\delta + (1+\nu_2+\eta^2-\eta^2\nu_2)\left(1+12\frac{h_0^2}{h_2^2}\right) - \\ - 12\frac{h_0^2}{h_2^2}\cdot\frac{(1+\nu_2+\eta^2-\eta^2\nu_2)^2}{(1+\nu_2+\eta^2-\eta^2\nu_2)+\beta_1\eta(\eta^2-1)}.$$

The annular plate shown schematically in Figure 57,b is reinforced by a perfectly rigid ring or core at the inner edge. Letting in (4.188) $\delta \to \infty$, we obtain*

$$\left.\begin{aligned} \gamma_2 - \gamma_2^0 &= \frac{(1-\eta^2)\delta_*\gamma_2^0 - 2\eta\gamma_1^0}{\eta^2(1+\nu+\delta_*)+(1-\nu-\delta_*)} \\ \gamma_1 &= u_1 = u_2 = 0 \end{aligned}\right\}. \tag{4.191}$$

The plate shown in Figure 57,b has no outer rib. The solution is obtained from (4.188) by setting $\delta_* = 0$, i.e., writing $-\alpha_{33}$ for α_5.

An annular plate is reinforced along the edges by ribs symmetrical to the middle plane, and simply supported along the circle $r = \varrho$ (Figure 58). The solution for this case is obtained from (4.171) by substituting, in addition to (4.187), also $h_0 = h_* = 0$ (the same result is obtained from (4.188) for $h_0 = 0$). We finally find that $u_1 = u_2 = 0$ and that in the expressions γ_1 and γ_2 in (4.188), Δ_8 has to be replaced by

$$\Delta_{11} = \alpha_5[\eta(\eta^2-1)\delta + (1+\nu_2+\eta^2-\eta^2\nu_2)] - 4\eta^2. \tag{4.192}$$

Bending stresses only act on the plate in this case. This problem was discussed in /31/ for certain cases of loading. If in addition, δ or δ_* vanishes, we obtain the solution for an annular plate reinforced, respectively, only along the outer or the inner edge by an elastic rib symmetrical to the middle plane (cf. §§ 5, 6 of Chapter III).

Circular plate with reinforced edge

A circular plate of uniform thickness, reinforced at the edge by a thin rib, is simply supported along the circle $r = \varrho$ (Figure 59) and loaded

* If w_2^0 is the deflection of a smooth annular plate clamped along the edge $r = R_1$ or $r = R$, we must set in (4.190) and (4.191), respectively, $\gamma_2^0 = 0$ or $\gamma_1^0 = 0$.

axisymmetrically. Setting $h_0 = \delta = \beta_1 = 0$, $\nu_1 = \nu_2 = \nu$, $\beta_3 = \lambda = 1$, $h_1 = h_2 = h$, and $\eta \to \infty$ in (4.171), we finally obtain

$$\left.\begin{aligned} &\gamma_1 = \gamma_1^0 = u_1 = 0, \qquad u_2 = \frac{(1+\nu)h_*}{1+\nu+\beta_2}\gamma_2 \\ &\gamma_2 = \frac{(1+\nu)(1+\nu+\beta_2)\gamma_2^0}{12\beta_2\frac{h_*^2}{h^2}(1+\nu)+(1+\nu+\delta_*)(1+\nu+\beta_2)} \end{aligned}\right\}. \tag{4.193}$$

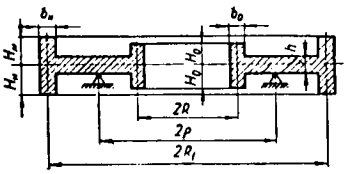


FIGURE 58.

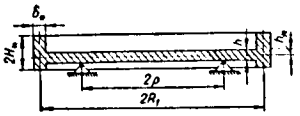


FIGURE 59.

If the plate is supported along the edge $r = R_1$, which is thus fixed ($\beta_2 = \infty$), we obtain

$$\left.\begin{aligned} &\gamma_2 = \frac{(1+\nu)\gamma_2^0}{\delta_* + (1+\nu)\left(1 + 12\frac{h_*^2}{h^2}\right)} \\ &\gamma_1 = u_1 = u_2 = 0 \end{aligned}\right\}. \tag{4.194}$$

If the stiffening rib is symmetrical to the middle plane of the plate ($h_* = 0$), then

$$u_1 = u_2 = \gamma_1 = 0, \qquad \gamma_2 = \frac{(1+\nu)\gamma_2^0}{\delta_* + 1 + \nu}. \tag{4.195}$$

Example. For the homogeneous plate of uniform thickness ($\nu_1 = \nu_2 = \nu = 0.3$, $h_1 = h_2 = h$) shown in Figure 55,b we have, for $\varrho = R_1$, $\beta_2 = \infty$, plotted the dimensionless magnitudes $\bar{w}_1$, $\bar{w}_2$, $\bar{\sigma}_1$, $\bar{\sigma}_{r2}$, $\bar{\sigma}_{\theta 2}$, $\bar{\sigma}_1^*$, $\bar{\sigma}_{r2}^*$, $\bar{\sigma}_{\theta 2}^*$ from (4.183) and (4.184) (Figures 60 to 67). These magnitudes do not depend on the load but only on the parameters $\frac{h_0}{h}$, δ, η, and $\frac{r}{R_1}$. They determine, except for known constant factors, the additional stresses and deflections due to the embossing, under an axisymmetrical load.

The embossing web (cf. Figure 51) is assumed to have a rectangular section with sides b and $2h_0$. According to (4.169a), we have

$$\beta_1 = \frac{EF(1-\nu_2^2)}{E_2 h_2 R_1} = \frac{h_2^2}{4h_0^2}\delta. \tag{4.196}$$

The curves in Figures 60 to 64 were plotted for different values of $\eta = \frac{R_1}{R}$, as functions of the parameter $\gamma_1^* = 1 - \frac{\gamma_1}{\gamma_1^0}$, which varies from 0 to 1 for $0 \leqslant \delta \leqslant \infty$ and $0 \leqslant \frac{h_0}{h} \leqslant \infty$.

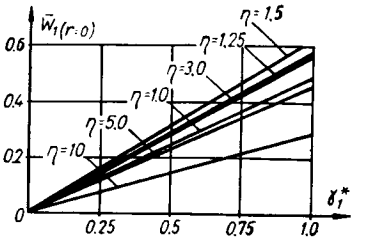


FIGURE 60.

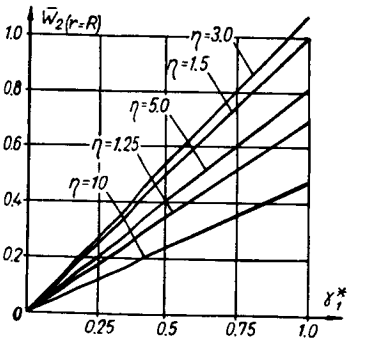


FIGURE 61.

Some values of γ_1^*, calculated by (4.184) for $\nu = 0.3$ as a function of $\frac{h_0}{h}$, δ, and η, are given in Table 18.

Figures 65 to 67 show curves of $\bar{\sigma}_1^*$, $(\bar{\sigma}_{r2}^*)_{r=R}$, and $(\sigma_{\theta 2}^*)_{r=R}$ for different values of η, as functions of the parameter β_1 ((4.196)) whose value is usually less than unity.

The magnitudes $\bar{\sigma}_{\theta 2}$, σ_{r2}^*, and $\bar{\sigma}_{\theta 2}^*$ are much smaller on the circle $r = R_1$ than on the circle $r = R$. The magnitudes, plotted in Figures 60 to 67, together with the basic stresses and deflections of the plate without embossing (which are assumed as known for given loads), determine by (4.183) the normal and bending stresses, and the deflections of an embossed plate.

The influence of the embossing on the stresses and deflections of the plate was studied in /166/, /32/, /33/, for two cases of loading.

It was shown that the embossing increases considerably the rigidity of the plate and reduces the calculated stresses. To each particular load there correpond an optimum radius and depth of the embossing.

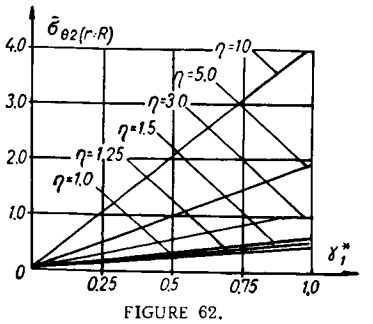


FIGURE 62.

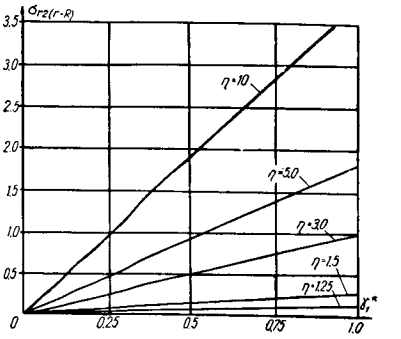


FIGURE 63.

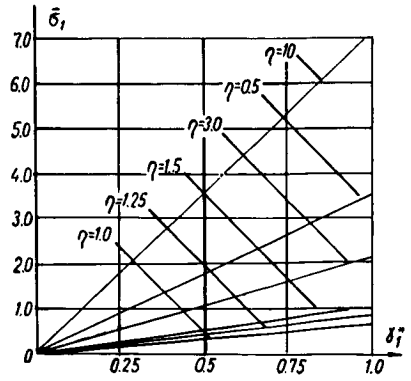


FIGURE 64.

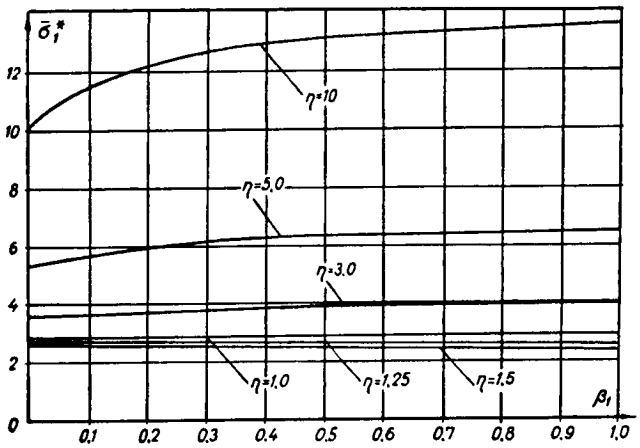


FIGURE 65.

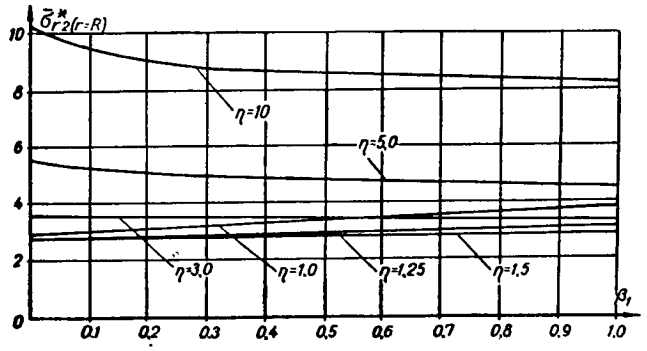


FIGURE 66.

TABLE 18

η	h_0/h					
	0.1	0.2	0.3	0.4	0.5	1.0
			$\delta=0.01$			
1.0	0.338	0.661	0.813	0.885	0.923	0.980
1.25	0.250	0.565	0.744	0.838	0.889	0.970
1.5	0.203	0.499	0.690	0.798	0.861	0.961
3.0	0.127	0.361	0.558	0.691	0.777	0.933
5.0	0.112	0.327	0.520	0.658	0.750	0.923
10	0.106	0.312	0.503	0.642	0.737	0.918
			$\delta=0.1$			
1.0	0.440	0.690	0.822	0.889	0.925	0.980
1.25	0.300	0.584	0.748	0.840	0.891	0.970
1.5	0.241	0.515	0.697	0.801	0.862	0.961
3.0	0.163	0.381	0.568	0.696	0.780	0.933
5.0	0.149	0.350	0.532	0.664	0.753	0.923
10	0.144	0.336	0.516	0.649	0.741	0.919
			$\delta=1.0$			
1.0	0.781	0.833	0.881	0.914	0.938	0.980
1.25	0.523	0.690	0.782	0.862	0.902	0.971
1.5	0.469	0.624	0.745	0.828	0.874	0.962
3.0	0.402	0.523	0.643	0.736	0.802	0.935
5.0	0.389	0.503	0.620	0.713	0.781	0.926
10	0.384	0.495	0.610	0.703	0.772	0.922

Let a uniformly distributed load q = const. act on the plate shown in Figure 55,b. For

$$\frac{b_0}{R_1}=\frac{1}{20}, \qquad \frac{h_0}{h}=1, \qquad \nu=0.3, \qquad \eta=1.5, \tag{4.197}$$

and $E_1=E_2=E$ we obtain from (4.170) and (4.196):

$$\delta=8\frac{b_0}{R_1}\left(\frac{h_0}{h}\right)^3(1-\nu^2)=0.364, \qquad \beta_1=0.091. \tag{4.198}$$

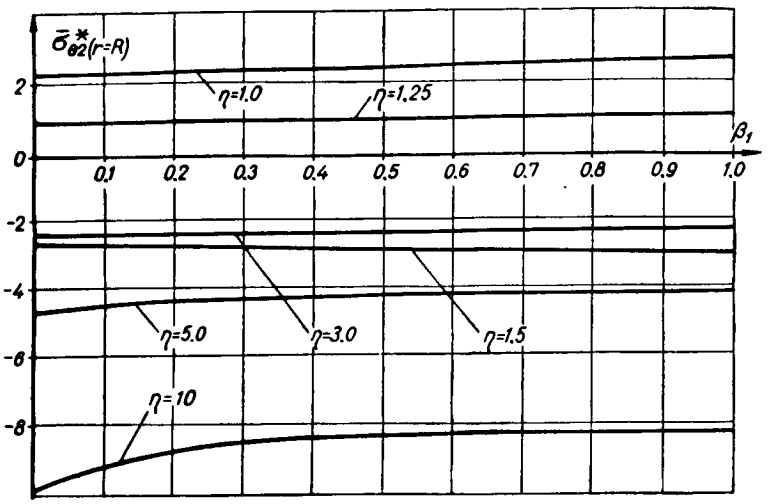


FIGURE 67

For this value of δ we obtain from Table 18 by interpolation:

$$\gamma_1^* = 0.961. \tag{4.199}$$

From the curves for $\eta = 1.5$ in Figures 60, 64, 65 we find the values of $\bar{w}_1|_{r=0}, \bar{\sigma}_1$, and $\bar{\sigma}_1^*$ corresponding to the values of γ_1^* ((4.199)) and β_1 ((4.198)).

$$\bar{w}_1|_{r=0} = 0.607, \quad \bar{\sigma}_1 = 1.029, \quad \bar{\sigma}_1^* = 2.716. \tag{4.200}$$

For a hinged circular plate without embossing, subjected to a uniform pressure q, we have (cf. /142/, p. 320)

$$w_1^0|_{r=0} = 0.0637\,\frac{qR_1^4}{D}, \quad \gamma_1^0 = -0.08725\,\frac{qR_1^3}{D}, \quad M_{r1}^0|_{r=0} = 0.20625\,qR_1^2. \tag{4.201}$$

Substitution of (4.200) and (4.201) in (4.183) yields:

maximum deflection at the center of the embossed plate:

$$w_1|_{r=0} = (0.0637 - 0.0530)\,\frac{qR_1^4}{D} = 0.0107\,\frac{qR_1^4}{D}; \tag{4.202}$$

maximum bending stresses at the same point:

$$\pm\,\sigma_{r1}|_{r=0} = \pm\,\sigma_{\theta 1}|_{r=0} = (1.2375 - 0.9804)\,\frac{qR_1^2}{h^2} = 0.2571\,\frac{qR_1^2}{h^2}; \tag{4.203}$$

normal stresses at the center of the plate:

$$\sigma_{r1}^* = \sigma_{\theta 1}^* = -\bar{\sigma}_1^*\left(\frac{h_0}{h}\right)\cdot 0.08725\cdot 0.039\,\frac{12}{h^2}\,(1-\nu^2)\,qR_1^2 = -0.1009\,\frac{qR_1^2}{h^2}. \tag{4.204}$$

Adding (4.203) and (4.204), we obtain the maximum total compressive stresses at the center of the embossed plate:

$$\sigma = -(0.2571 + 0.1009)\frac{qR_1^2}{h^2} = -0.358\frac{qR_1^2}{h^2}. \tag{4.205}$$

We give below the values of the maximum deflection and stresses at the center of a homogeneous circular nonembossed plate, hinged at the edge (Figure 12). The plate is reinforced at $r = \frac{R_1}{1.5}$ by a rib of the same material and the same rigidity $\delta = 0.364$ (4.198), located symmetrical to the middle plane (cf. § 12 of Chapter II):

$$\left.\begin{aligned} w_1|_{r=0} &= (0.0637 - 0.0139)\frac{qR_1^4}{D} = 0.0498\frac{qR_1^4}{D} \\ \pm\sigma &= (1.2375 - 0.2579)\frac{qR_1^2}{h^2} = 0.9796\frac{qR_1^2}{h^2} \end{aligned}\right\} \tag{4.206}$$

Comparing (4.206) with (4.202) and (4.205), we find that the ratio between the maximum deflections of the two plates is 1 : 4.65, that of the stresses at the center being 1 : 2.74. This shows that the embossing increases both rigidity and strength of the plate.

Chapter V

INVERSE PROBLEMS OF PLATES WITH HOLES WHOSE EDGES ARE REINFORCED BY THIN RIBS

In Chapters III and IV we discussed the direct problem of plates reinforced around holes, which consists in determining the state of stress of the system when the shape of the hole and the flexural and torsional rigidities of the stiffening rib are known.

In this chapter we consider some inverse problems.

§ 1. FIRST INVERSE PROBLEM*

An isotropic or anisotropic thin plate of finite dimensions has one or several holes or grooves. The edges of the holes or grooves are reinforced by thin elastic isotropic ribs, whose flexural (A) and torsional (C) rigidities are in the general case functions of the arc length s measured along the axis Γ of the rib. One of the principal axes of inertia of the cross sections of each rib lies in the middle plane of the plate. The x- and y-axes lie in the middle plane, while the Z-axis points downward.

We define an elastic stiffening rib or bar system as equivalent if, at a given load on the plate, it fully compensates the missing part of the plate, i.e., if the condition

$$w(x, y) \equiv w^0(x, y), \tag{5.1}$$

is satisfied, where $w^0(x, y)$ is the basic deflection of the middle plane of the solid (nongrooved) plate, and $w(x, y)$ is the deflection of the same plate under the same load, but with a hole (groove) whose edge is reinforced by an equivalent rib.

Hence, an equivalent reinforcing system completely prevents stress concentrations near the holes or grooves in the plate.

The first inverse problem consists in determining the flexural (A) and torsional (C) rigidities of an equivalent bar system, if the basic deflection w^0 and the shape of the holes (grooves) are specified.

Consider first a plate having holes with smooth edges. It is obvious from (5.1) that, without loss of generality, we may consider a plate with

* The first solution of this problem is due to N. P. Fleishman, who presented it at the XXII-th Scientific Session of the Leningrad State University (19 March 1953), and published it in /146/ and /147/. A similar problem in the two-dimensional theory of elasticity was later solved independently by Mansfield (/208, 209/). Papers by Mossakovskii and Kvashi /80/, and by Kvashi /49, 50/ deal with the selection of neutral reinforcements around holes in shells of revolution.

only one hole (Γ), since other holes, whose edges are reinforced by equivalent ribs, do not affect the basic states of stress and strain of the plate.

We shall assume the function $w^0(x, y)$ to be known. Hence, the shearing forces N_n, the bending moments M_n, the twisting moments $H_{n\tau}$, and the deflection derivatives $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial n}$ are known everywhere in the solid plate and by (5.1) also in the reinforced plate, in particular on Γ. The symbols n and τ represent, respectively, the principal normal and the tangent to Γ. The positive direction in which the arc length s is measured is such that τ and n form a system similar to the x- and y-axes.

The conditions at the joint between the equivalent rib and the plate are

$$p(s) = N_n + \frac{\partial H_{n\tau}}{\partial s}, \qquad m(s) = M_n, \tag{5.2}$$

$$\theta_\tau(s) = \frac{\partial w}{\partial n}, \qquad \theta_n(s) = -\frac{\partial w}{\partial s}, \tag{5.3}$$

where $p(s)$ and $m(s)$ are, respectively, the shearing force and bending moment with which the plate acts on the rib; $\theta_\tau(s)$ is the angle of twist of the rib; $\theta_n(s)$ is the angle of bending of the rib.

The right-hand sides of (5.2) and (5.3) are known. These conditions state that action and reaction between rib and plate are equal, as are the angles of rotation of the radial and tangential sections of rib and plate along Γ. We have thus to determine the flexural and torsional rigidities $A(s)$ and $C(s)$ of a thin elastic bar forming a plane closed curve, when the load is given by (5.2) and the strains by (5.3).

The internal moment $\vec{L}$, acting in section s of the rib, is obtained by integrating the equations of statics for a rib element* (Figure 68):

$$\vec{L} = -\vec{L}_0 - [\vec{r}(s) - \vec{r}_0] \times N_0\vec{k} - \int_0^s \vec{\tau}(\sigma)\, m(\sigma)\, d\sigma + \int_0^s [\vec{r}(s) - \vec{r}(\sigma)] \times p(\sigma)\vec{k}\, d\sigma, \tag{5.4}$$

where $\vec{k}, \vec{\tau}$ are, respectively, unit vectors in the z and τ directions; $\vec{L}_0$ and N_0 are arbitrary integration constants; $\vec{r}(s)$ is the radius-vector of point s. We shall use the following approximate relationships (cf. (2.10)):

$$\begin{aligned} L_n &= A(s)\,\delta\omega_n, \\ L_\tau &= C(s)\,\delta\omega_\tau, \end{aligned} \tag{5.5}$$

where L_n and L_τ are, respectively, the internal bending and twisting moments, being the projections of the moment $\vec{L}$ on n and τ; $\delta\omega_n$ and $\delta\omega_\tau$ are the changes in curvature and torsion of the rib, determined by Clebsch's formulas (2.14):

$$\delta\omega_n = \frac{d}{ds}\theta_n + q\theta_\tau, \qquad \delta\omega_\tau = \frac{d}{ds}\theta_\tau - q\theta_n. \tag{5.6}$$

* When considering the equilibrium of the rib under the action of a load imposed by the plate, the twisting couples should better not be replaced by equivalent transverse forces. The authors realized this only after the book had already been set.

Here $q = q(s)$ is the variable curvature of the rib axis Γ. Substituting (5.6) in (5.5) and using (5.3), we obtain for the case when Γ is a closed smooth contour:

$$A(s) = \frac{L_n}{\left[q\frac{\partial w}{\partial n} - \frac{\partial^2 w}{\partial s^2}\right]_\Gamma}, \qquad C(s) = \frac{L_\tau}{\left[q\frac{\partial w}{\partial s} + \frac{\partial^2 w}{\partial n \partial s}\right]_\Gamma}. \tag{5.7}$$

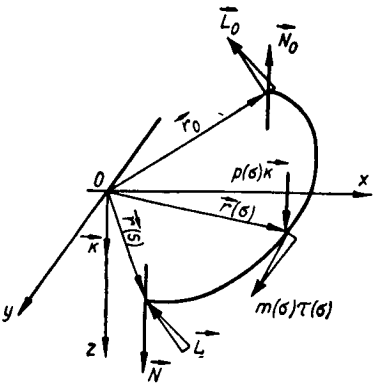


FIGURE 68.

Since N_0 and $\vec{L}_0$ are arbitrary, the above solution is not unique; in fact, by assigning other values to these constants, we obtain different rib rigidities and internal forces at equal strains due to the same loads.

Furthermore, a solution does not always exist, since by their physical meaning, the functions $A(s)$ and $C(s)$ must have finite and positive values.

The periodicity condition for the strains, rigidities, and internal forces of the rib are satisfied for any values of N_0 and $\vec{L}_0$.

We conclude on the basis of (5.1), (5.2), and (5.3), that the elastic strain energy of a rib, whose rigidities are given by (5.7), equals the elastic strain energy of the part of the plate replaced by it.

If the holes in the plate have piecewise-smooth edges, (5.7) can be applied separately to each of their parts. At the corners the relevant compatibility conditions have to be satisfied (cf. §§ 3 to 6 of this chapter).

When considering the equivalent reinforcement of groove edges, we must satisfy the condition that no moments or shearing forces act on the end sections of the stiffening bar, as will be shown below (cf. § 7).

§ 2. EQUIVALENT REINFORCEMENT OF A CIRCULAR HOLE EDGE

Consider a circular hole of radius R. We project the moment vector (5.4) on the n and τ axes. Introducing polar coordinates (r, θ) with origin at

the center of the hole, we obtain

$$\left.\begin{aligned} L_r &= L_{0r}\cos\theta - (RN_0 - L_{0\tau})\sin\theta - R\int_0^\theta [m(\alpha) - \\ &- Rp(\alpha)]\sin(\theta-\alpha)\,d\alpha \\ L_\theta &= -L_{0r}\sin\theta + RN_0 + (L_{0\tau} - RN_0)\cos\theta + \\ &+ R\int_0^\theta [Rp(\alpha) - m(\alpha)]\cos(\theta-\alpha)\,d\alpha - R^2\int_0^\theta p(\alpha)\,d\alpha \end{aligned}\right\}. \tag{5.8}$$

Substituting in (5.7) $q = \frac{1}{R} = \text{const.}$, we obtain

$$A(\theta) = \frac{L_r}{\left[\frac{1}{R}\cdot\frac{\partial w}{\partial r} + \frac{1}{R^2}\cdot\frac{\partial^2 w}{\partial\theta^2}\right]_\Gamma}, \qquad C(\theta) = \frac{L_\theta}{\left[\frac{1}{R^2}\cdot\frac{\partial w}{\partial\theta} - \frac{1}{R}\cdot\frac{\partial^2 w}{\partial r\partial\theta}\right]_\Gamma}. \tag{5.9}$$

Let us consider some examples.

a) A circular cylindrical anisotropic (orthotropic) plate with a hole in the center is loaded symmetrically to its center which forms the pole of anisotropy. The central part of the solid plate would have been subjected to pure bending, the deflections being (/ 64/, § 54)

$$w^0 = Br^{1+\beta} + \text{const.} \tag{5.10}$$

where $\beta = \sqrt{\frac{D_\theta}{D_r}}$; D_r and D_θ are the flexural rigidities of the plate referred the θ and r axes at the point considered; B is a constant.

We have

$$\begin{aligned} m(\theta) &= -D_r\left(w'' + \frac{\nu_\theta}{r}w'\right)_\Gamma = -D_r BR^{\beta-1}(1+\beta)(\beta+\nu_\theta) = m, \\ p(\theta) &= -D_r\left(w''' + \frac{1}{r}w''\right) + D_\theta\frac{1}{r^2}w' = 0, \end{aligned} \tag{5.11}$$

where ν_θ is one of the principal Poisson ratios.

Substituting (5.11) in (5.8), we find

$$\begin{aligned} L_r &= L_{0r}\cos\theta - (RN_0 - L_{0\theta})\sin\theta + mR(\cos\theta - 1), \\ L_\theta &= -L_{0r}\sin\theta + RN_0 + (L_{0\theta} - RN_0)\cos\theta - mR\sin\theta. \end{aligned}$$

For the state of stress of the rib to be symmetrical to the center, we require that

$$L_{0r} = -mR,\ L_{0\theta} = RN_0,\ N_0 = 0.$$

We then obtain

$$L_r = -mR,\ L_\theta = 0. \tag{5.12}$$

Substitution of (5.10) and (5.12) in (5.9) yields

$$A = RD_r(\beta + \nu_\theta). \tag{5.13}$$

$C = \frac{0}{0}$ is arbitrary.

Thus, in the case considered, an equivalent rib exists. Its flexural rigidity is constant, being given by (5.13), while its torsional rigidity does not affect the solution.

In the particular case of an isotropic plate $D_\theta = D_r = D$, $\nu_\theta = \nu$, $\beta = 1$, and we obtain from (5.13)

$$A = RD(1 + \nu). \tag{5.14}$$

This same result was obtained in §§ 3 and 4 of Chapter III by other methods for certain cases of axisymmetrical bending of an isotropic plate.

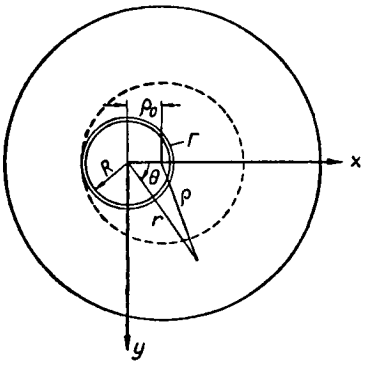


FIGURE 69.

b) A circular isotropic plate is loaded symmetrically to its center and has an eccentric hole (Figure 69). No loads act on the circular area concentric with the plate, which encloses the hole. If the plate were solid, its central load-free part would be in a state of pure bending, its deflections being

$$w^0 = B\varrho^2 + \text{const},$$

where ϱ is the distance between points of the middle plane and the center of the plate. Introducing polar coordinates (r, θ) with origin at the center of the eccentric hole, we obtain

$$w^0 = B(r^2 + \varrho_0^2 + 2r\varrho_0 \cos\theta) + \text{const},$$

where ϱ_0 is the eccentricity of the hole; B is a constant.

We have, according to (1.16) and (1.55),

$$\left.\begin{aligned} m(\theta) &= -D\left(\frac{\partial^2 w}{\partial r^2}+\frac{\nu}{r}\cdot\frac{\partial w}{\partial r}+\frac{\nu}{r^2}\cdot\frac{\partial^2 w}{\partial\theta^2}\right)_\Gamma = -2BD(1+\nu)\\ p(\theta) &= -D\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial w}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 w}{\partial\theta^2}\right)_\Gamma -\\ &\quad -D(1-\nu)\frac{1}{R}\cdot\frac{\partial}{\partial\theta}\left(\frac{1}{r}\cdot\frac{\partial^2 w}{\partial r\partial\theta}-\frac{1}{r^2}\cdot\frac{\partial w}{\partial\theta}\right)_\Gamma = 0 \end{aligned}\right\} \qquad (5.15)$$

Substituting (5.15) in (5.8), and then in (5.9), and selecting the arbitrary constants as before, we obtain the flexural rigidity of the equivalent rib

$$A = RD(1+\nu). \qquad (5.16)$$

The torsional rigidity of the rib is arbitrary. As ought to be expected, the equivalent ribs are the same for equal holes, regardless of their number and positions.

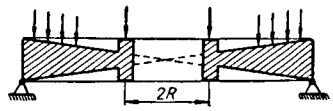


FIGURE 70.

c) A circular isotropic plate of variable thickness, with central hole of radius R, is bent symmetrically (Figure 70). The rigidity variation of the plate is given by $D = D_0 r^m$, where $D_0 =$ const. and $m =$ const. If the plate were solid, its central load-free part would be subjected to bending moments uniformly distributed along the edge. The basic deflections of the solid plate are (/ 55/, p. 68)

$$w^0 = -\frac{C_0}{\alpha-\frac{m}{2}+1}r^{\alpha-\frac{m}{2}+1} + \text{const} \quad (r \leqslant R),$$

where

$$C_0 = \text{const}, \quad \alpha = \sqrt{1-m\nu+\frac{m^2}{4}}, \quad m < \frac{1}{\nu}.$$

Thus,

$$\left.\begin{aligned} &\frac{\partial w^0}{\partial\theta} = \frac{\partial^2 w^0}{\partial r\partial\theta} = \frac{\partial^2 w^0}{\partial\theta^2} = 0, \quad \frac{\partial w^0}{\partial r} = -C_0 r^{\alpha-\frac{m}{2}}\\ &m(\theta) = (M_r)_\Gamma = D_0 C_0\left(\alpha-\frac{m}{2}+\nu\right)R^{\alpha-\frac{m}{2}+\alpha-1}\\ &p(\theta) = 0, \quad L_\theta = 0, \quad L_r = -Rm(\theta) \end{aligned}\right\}. \qquad (5.17)$$

Substituting (5.17) in (5.9), we obtain the rigidities of the equivalent rib

$$A = D_0 R\left(\alpha - \frac{m}{2} + \nu\right),\ C = \frac{0}{0} \text{ is arbitrary.}$$

In the particular case of a plate whose thickness varies linearly, we have $m = 3$; for $\nu = \frac{1}{3}$, we obtain

$$A = \frac{1}{3} R (D)_{r=R}. \tag{5.18}$$

In this example the equivalent rib replaces a disk of variable thickness.

d) A bounded rectangular isotropic plate is in a homogeneous state of stress. The basic deflections are then

$$w^0 = ax^2 + bxy + cy^2, \tag{5.19}$$

where a, b, c are constants.

If in this plate we make a circular hole of radius R and reinforce its edge by an equivalent rib, the load on the rib becomes, regardless of the position of the hole,

$$\left.\begin{aligned} m(\theta) &= (M_r)_\Gamma = -D\{(1+\nu)(a+c) + (1-\nu)[(a-c)\cos 2\theta + b\sin 2\theta]\} \\ p(\theta) &= \left(N_r + \frac{1}{R}\cdot\frac{\partial H_{r\theta}}{\partial \theta}\right)_\Gamma = \frac{2}{R} D(1-\nu)[(a-c)\cos 2\theta + b\sin 2\theta] \end{aligned}\right\}. \tag{5.20}$$

Substituting (5.20) in (5.8), we obtain

$$\begin{aligned} L_r = RD[(c-a)(1-\nu)\cos 2\theta + (c+a)(1+\nu) - b(1-\nu)\sin 2\theta] + \\ + C_1\cos\theta + C_2\sin\theta, \\ L_\theta = RD[(a-c)(1-\nu)\sin 2\theta - b(1-\nu)\cos 2\theta] + C_3 - \\ - C_1\sin\theta + C_2\cos\theta, \end{aligned} \tag{5.21}$$

where

$$C_1 = L_{0r} - 2RD(c + a\nu),$$
$$C_2 = L_{0\theta} - N_0 R + 2bRD(1-\nu),\ C_3 = N_0 R - b(1-\nu)RD.$$

Differentiating (5.19) with respect to r and θ yields

$$\left.\begin{aligned} \left(\frac{1}{R^2}\cdot\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{R}\cdot\frac{\partial w}{\partial r}\right)_\Gamma &= (c-a)\cos 2\theta + (a+c) - b\sin 2\theta \\ \left(\frac{1}{R^2}\cdot\frac{\partial w}{\partial\theta} - \frac{1}{R}\cdot\frac{\partial^2 w}{\partial r\partial\theta}\right)_\Gamma &= (a-c)\sin 2\theta - b\cos 2\theta \end{aligned}\right\}. \tag{5.22}$$

Substituting (5.21) and (5.22) in (5.9), and setting $C_1 = C_2 = C_3 = 0$, we obtain

$$\left.\begin{aligned} A &= RD\frac{(c-a)(1-\nu)\cos 2\theta + (c+a)(1+\nu) - b(1-\nu)\sin 2\theta}{(c-a)\cos 2\theta + (c+a) - b\sin 2\theta} \\ C &= RD(1-\nu) \end{aligned}\right\}. \tag{5.23}$$

Equation (5.23) shows that in this case the equivalent rib must generally have a variable rigidity. The coefficients a, b, c are given in Table 19 for two different kinds of load acting on the plate.

TABLE 19

Kind of load	a	b	c
Bending by moments M_x and M_y acting on the edges	$\frac{\nu M_y - M_x}{2D(1-\nu^2)}$	0	$\frac{\nu M_x - M_y}{2D(1-\nu^2)}$
Torsion by twisting moments H= = const. distributed along the edges	0	$\frac{H}{D(\nu-1)}$	0

If the plate undergoes pure bending, i. e., $M_x = M_y = M$ and $H = 0$, we obtain from (5.23) $A = RD(1+\nu)$ while C is arbitrary. This coincides with (5.16), as ought to be expected.

If the plate undergoes pure torsion, i. e., $M_x = M_y = 0$, then

$$A = RD(1-\nu), \quad C = RD(1-\nu). \tag{5.24}$$

This result was obtained by a different procedure in § 3 of Chapter III.

In the case of combined bending and torsion, i. e., for $M_x = M_y = M$ and $H \neq 0$, we have

$$\left.\begin{aligned} A &= RD\frac{(1-\nu^2)(H\sin 2\theta - M)}{(1+\nu)H\sin 2\theta - (1-\nu)M} \\ C &= RD(1-\nu) \end{aligned}\right\}. \tag{5.25}$$

In this case there might be no equivalent rib. Since the rigidity A ((5.25)) is positive and finite, we must have

$$\left|\frac{H}{M}\right| < \frac{1-\nu}{1+\nu}.$$

If the plate is subjected to bending moments M_x and $M_y = \eta M_x$ acting on the edges while $H = 0$, an equivalent rib exists only when

$$\nu < \eta < \frac{1}{\nu}. \tag{5.26}$$

This shows that when bending moments act in a single sense ($\eta = 0$), and also in the case of cylindrical bending $\left(\eta = \frac{1}{\nu} \text{ or } \eta = \nu\right)$ no equivalent rib exists.

Finally, when $M_x = -M_y$, i. e., $\eta = -1$ and $[H = 0]$, we obtain from (5.23) $A = C = RD(1-\nu)$, as ought to be expected according to (5.24).

From (5.13), (5.16), (5.23), we obtain the dimensions of the equivalent rib by the method described in § 10 of Chapter III.

As an example, we shall determine the dimensions of the equivalent rib, of variable rectangular cross section, for a plate subjected to bending

moments acting on the edges (Figure 71), with $\eta = \frac{1}{2}$, $\nu = 0.3$.

The rigidities of the circular rib are in this case by (5.23):

$$A = RD(1-\nu^2)\frac{(1-\eta)\cos 2\theta - (1+\eta)}{(1+\nu)(1-\eta)\cos 2\theta - (1-\nu)(1+\eta)}, \qquad (5.27)$$

$$C = RD(1-\nu).$$

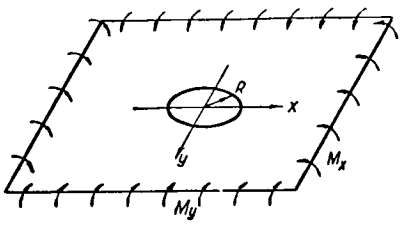


FIGURE 71.

Denoting by k the ratio $\frac{h_1}{b}$ between height and width of the rib section, we obtain for a rectangular section

$$A = \frac{E_1 k^3 b^4}{12}, \quad C = \frac{E_1 \alpha k b^4}{2(1+\nu_1)}. \qquad (5.28)$$

Here E_1 is Young's modulus and ν_1 is Poisson's ratio for the rib; the coefficient α is determined from the known solution of the problem of torsion of a prismatic bar having a rectangular section /23/. Values of α for various k are given in Table 20.

TABLE 20

k	α	k	α	k	α
1	0,14057	1,35	0,18210	1,70	0,21093
1,05	0,14744	1,40	0,18689	1,75	0,21425
1,10	0,15398	1,45	0,19144	1,80	0,21742
1,15	0,16019	1,50	0,19575	1,85	0,22044
1,20	0,16612	1,55	0,19985	1,90	0,22332
1,25	0,17173	1,60	0,20373	1,95	0,22606
1,30	0,17705	1,65	0,20742	2,00	0,22868

For $\nu_1 = \nu = 0.3$ we obtain from (5.27) and (5.28)

$$\frac{A}{C} = \frac{1.3}{6}\cdot\frac{k^2}{\alpha} = 1.3\frac{3-\cos 2\theta}{2.1-1.3\cos 2\theta}. \qquad (5.29)$$

We can determine $k = \frac{h_1}{b}$ as a function of the polar angle θ from (5.29) and Table 20. Equations (5.28) and (5.27) then yield formulas for the

relative dimensions of the equivalent rib:

$$\frac{b}{h}=\sqrt[4]{\frac{E}{E_1}\cdot\frac{R}{h}}\sqrt[4]{\frac{1}{k^3}\cdot\frac{3-\cos 2\theta}{2.1-1.3\cos 2\theta}},$$
$$\frac{h_1}{h}=\sqrt[4]{\frac{E}{E_1}\cdot\frac{R}{h}}k\sqrt[4]{\frac{1}{k^3}\cdot\frac{3-\cos 2\theta}{2.1-1.3\cos 2\theta}}. \tag{5.30}$$

Table 21 gives values of b/h and h_1/h for the equivalent rib reinforcing the edge of a circular hole in a rectangular plate subjected to bending moments M_x and $M_y=0.5M_x$ (Figure 71).

TABLE 21

θ°	$\frac{b}{h}\sqrt[4]{\frac{E_1}{E}\cdot\frac{h}{R}}$	$\frac{h_1}{h}\sqrt[4]{\frac{E_1}{E}\cdot\frac{h}{R}}$	θ°	$\frac{b}{h}\sqrt[4]{\frac{E_1}{E}\cdot\frac{h}{R}}$	$\frac{h_1}{h}\sqrt[4]{\frac{E_1}{E}\cdot\frac{h}{R}}$
0	0.806	1.458	50	0.979	1.117
5	0.810	1.448	55	0.995	1.097
10	0.820	1.420	60	1.009	1.081
15	0.835	1.380	65	1.021	1.068
20	0.854	1.333	70	1.031	1.057
25	0.875	1.287	75	1.037	1.048
30	0.897	1.243	80	1.043	1.043
35	0.919	1.204	85	1.046	1.040
40	0.941	1.170	90	1.047	1.039
45	0.961	1.141			

§ 3. HOLE-EDGE REINFORCEMENT OF A RECTANGULAR PLATE SUBJECTED TO BENDING MOMENTS ACTING IN TWO SENSES

A rectangular isotropic plate of finite dimensions is bent by applying at the edges uniformly distributed moments M_x and M_y. The hole in the plate is bounded by the arcs of two intersecting circles of radii R_1 and R_2. The location of the hole is arbitrary (Figure 72). The x- and y-axes are parallel to the axes of symmetry of the plate. It is required to solve the first inverse problem (§ 1 of this chapter).

The deflection function has in this case the form /118/

$$w=ax^2+cy^2, \tag{5.31}$$

where

$$a=\frac{\nu M_y-M_x}{2D(1-\nu^2)};$$
$$c=\frac{\nu M_x-M_y}{2D(1-\nu^2)}; \tag{5.32}$$

D is the flexural rigidity of the plate; ν is Poisson's ratio.

We introduce two polar coordinate systems (r_i, θ_i) $(i=1, 2)$ with origins, respectively, at the centers O_1, O of the circles. The polar axes are

parallel to the x-axis. The bending moments $m(R_i, \theta_i)$ and the shearing forces $p(R_i, \theta_i)$ with which the plate acts on the equivalent rib BB_iB' are obtained from (5.20) by setting $b = 0$:

$$m(R_i, \theta_i) = -D[(1+\nu)(a+c) + (1-\nu)(a-c)\cos 2\theta_i], \tag{5.33}$$

$$p(R_i, \theta_i) = \frac{2}{R_i} D(1-\nu)(a-c)\cos 2\theta_i. \tag{5.34}$$

In addition, the plate acts on the reinforcing ribs at B and B' with two concentrated forces each, normal to the xy plane. These forces are caused by the twisting moments $H_{r\theta}$ acting at the ends of the intersecting arcs. The forces acting on B and B' from the right are, respectively,

$$\left.\begin{aligned} P_1 &= (H_{r\theta})_B = (1-\nu)(a-c)D\sin 2(\varphi_1 - \varphi_0) \\ P_1' &= -(H_{r\theta})_{B'} = (1-\nu)(a-c)D\sin 2(\varphi_1 + \varphi_0) \end{aligned}\right\}, \tag{5.35}$$

and from the left

$$\left.\begin{aligned} P_2 &= -(H_{r\theta})_B = (1-\nu)(a-c)D\sin 2(\varphi_2 + \varphi_0) \\ P_2' &= (H_{r\theta})_{B'} = (1-\nu)(a-c)D\sin 2(\varphi_2 - \varphi_0) \end{aligned}\right\}. \tag{5.36}$$

The positive direction of the forces P_i, P_i' $(i = 1, 2)$ is here downward (along the Z-axis). The bending and twisting moments L_{ri} and $L_{\theta i}$ $(i = 1, 2)$, acting in the sections of the i-th rib BB_iB', are obtained from (5.21) by setting $b = C_1 = C_2 = C_3 = 0$

$$\left.\begin{aligned} L_{ri} &= R_iD[(c-a)(1-\nu)\cos 2\theta_i + (c+a)(1+\nu)] \\ L_{\theta i} &= R_iD(a-c)(1-\nu)\sin 2\theta_i \end{aligned}\right\}. \tag{5.37}$$

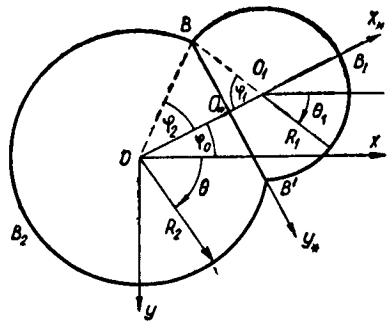


FIGURE 72.

In addition to the moments (5.37) the shearing forces

$$N_i = -R_i\int_0^{\theta_i} p(R_i, \beta_i)\,d\beta_i = -D(1-\nu)(a-c)\sin 2\theta_i. \tag{5.38}$$

act on the stiffening ribs.

In fact, if we add to (5.38) the unbalanced force (5.35) or (5.36) due to the twisting moment $H_{r\theta}$, the total shearing force vanishes, as will be proved later.

Substituting (5.31) and (5.37) in (5.7), we obtain the flexural and torsional rigidities of the curved bars of the equivalent system:

$$\left.\begin{aligned} A_i &= R_i D \frac{(1-\eta)\cos 2\theta_i - (1+\eta)}{(1+\nu)(1-\eta)\cos 2\theta_i - (1-\nu)(1+\eta)}(1-\nu^2) \\ C_i &= R_i D(1-\nu) \end{aligned}\right\}, \tag{5.39}$$

where $\eta = M_y/M_x$.

A necessary and sufficient condition for A_i to be finite and positive is

$$\nu < \eta < \frac{1}{\nu}. \tag{5.40}$$

The cross-sectional dimensions of the curved ribs having the rigidities (5.39) are determined as explained in the preceding section. These dimensions are generally variable, except when $\eta = \pm 1$. When $\eta = -1$, i.e., for $M_x = -M_y$, we have

$$A_i = C_i = R_i D(1-\nu). \tag{5.41}$$

The case $\eta = 1$ will be considered in detail in the next section.

Equations (5.37) and (5.38) show that the equivalent reinforcing system is in equilibrium if the following bending moments, twisting moments, and shearing forces act from the outside at B' and B:

$$\left.\begin{aligned} L'_{\theta i} &= -R_i D(a-c)(1-\nu)\sin 2(\varphi_0 \pm \varphi_i) \\ L'_{ri} &= R_i D[(c-a)(1-\nu)\cos 2(\varphi_0 \pm \varphi_i) + (c+a)(1+\nu)] \\ L_{\theta i} &= -R_i D(a-c)(1-\nu)\sin 2(\varphi_0 \mp \varphi_i) \\ L_{ri} &= R_i D[(c-a)(1-\nu)\cos 2(\varphi_0 \mp \varphi_i) + (c+a)(1+\nu)] \end{aligned}\right\}, \tag{5.42}$$

$$\left.\begin{aligned} N'_i &= D(1-\nu)(a-c)\sin 2(\varphi_0 \pm \varphi_i) \\ N_i &= D(1-\nu)(a-c)\sin 2(\varphi_0 \mp \varphi_i) \end{aligned}\right\}. \tag{5.43}$$

Here and below the upper sign corresponds to $i = 1$, and the lower sign to $i = 2$. The positive directions of moments (5.42) are shown in Figure 73. The forces N_1 and N'_2 act upward, and the forces N'_1 and N_2 downward.

The forces (5.35) and (5.36) act at B and B' in addition to the forces (5.43) and moments (5.42). The principal force vector P_z, and the projections M_u and M_k on the x_*- and y_*-axes (Figure 72) of the moments applied at point B are

$$\left.\begin{aligned} P_z &= P_1 + P_2 - N_2 + N_1 = 0 \\ M_u &= L_{r1}\cos\varphi_1 + L_{r2}\cos\varphi_2 + L_{\theta 2}\sin\varphi_2 - L_{\theta 1}\sin\varphi_1 = \\ &= D\varrho[(c-a)(1-\nu)\cos 2\varphi_0 + (c+a)(1+\nu)] \\ M_k &= L_{r1}\sin\varphi_1 + L_{\theta 1}\cos\varphi_1 - L_{r2}\sin\varphi_2 + L_{\theta 2}\cos\varphi_2 = \\ &= D\varrho(c-a)(1-\nu)\sin 2\varphi_0 \end{aligned}\right\}, \tag{5.44}$$

where ϱ is the distance between O and O_1.

We similarly find for point B'

$$P'_z = 0, \quad M'_u = -M_u, \quad M'_k = -M_k. \tag{5.45}$$

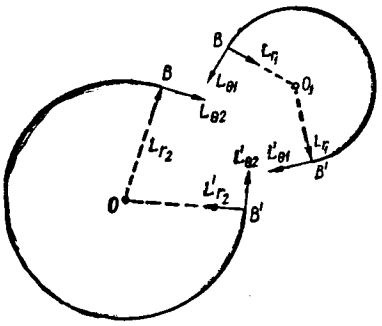


FIGURE 73.

The moments M_u, M_k and M'_u, M'_k must be applied to B and B' from the outside. A rectangular bar is therefore welded-in between these points. This bar is subjected to twisting moments M_k, M'_k and bending moments M_u, M'_u. The so far unknown flexural and torsional rigidities A^*, C^* of this bar are determined from the condition that the strains and deflections of plate and bar at B and B' be equal. At $x_* = 0, y_* = \pm d$

$$\frac{\partial w_*}{\partial y_*} = \frac{\partial w}{\partial y_*}, \tag{5.46}$$

$$\left.\frac{\partial w}{\partial x_*}\right|_B - \left.\frac{\partial w}{\partial x_*}\right|_{B'} = \varphi_*, \tag{5.47}$$

$$w_* = w, \tag{5.48}$$

where w_* is the deflection of bar BB'; φ_* is the relative rotation of its section at B and B'; $2d = BB'$.

We have

$$w_* = \frac{M_u}{2A^*} y_*^2 + C' y_* + C'', \tag{5.49}$$

where C' and C'' are integration constants to be determined.

Substituting (5.49) and (5.31) in (5.46), we obtain two equations from which we find:

$$A^* = \frac{1}{2} D\varrho (1 - \nu^2) \frac{(1 - \eta)\cos 2\varphi_0 - (1 + \eta)}{(\eta\nu - 1)\sin^2 \varphi_0 + (\nu - \eta)\cos^2 \varphi_0}, \tag{5.50}$$

$$C' = (a - c) R_2 \cos \varphi_2 \sin 2\varphi_0. \tag{5.51}$$

It is easily shown that A^* is positive and finite for any φ_0 and $\nu < \eta < \nu^{-1}$ [cf. (5.26)].

We determine C'' by substituting (5.49) and (5.31) in (5.48).

The angle of twist φ_* is

$$\varphi_* = \frac{2M_k d}{C^*}. \tag{5.52}$$

Substituting (5.52) and (5.31) in (5.47), we find the torsional rigidity of the bar:

$$C^* = \varrho D(1-\nu). \tag{5.53}$$

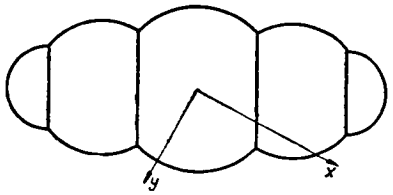


FIGURE 74.

The first inverse problem has thus been solved for the case considered. It is interesting to note that C^* is a constant and, unlike A^*, is independent of the ratio of the bending moments acting on the plate and of the orientation of the hole. If $\eta = 1$ (pure bending) or $\varphi_0 = 0$ or $\varphi_0 = \frac{\pi}{2}$, C^* remains arbitrary, since in these cases the twisting moment M_k (5.44) vanishes, and the bar BB' is not subjected to torsion. We obtain from (5.50):

$$\left.\begin{array}{ll} \text{for } \varphi_0 = 0 & A^* = \varrho D \dfrac{\eta(1-\nu^2)}{\eta - \nu} \\ \text{for } \varphi_0 = \dfrac{\pi}{2} & A^* = \varrho D \dfrac{1-\nu^2}{1-\nu\eta} \end{array}\right\}. \tag{5.54}$$

If $\varphi_0 = \frac{\pi}{4}$, then

$$A^* = \varrho D(1+\nu). \tag{5.55}$$

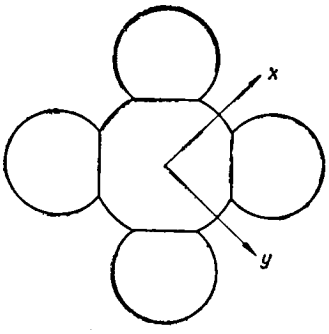


FIGURE 75.

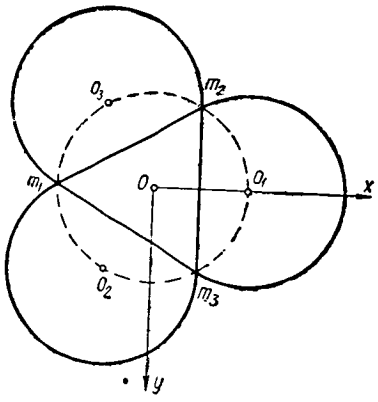


FIGURE 76.

The formulas obtained for the rigidities A_i, C_i((5.39)).A^*((5.50)), and C^*((5 53)) remain valid also for a hole bounded by arcs of several intersecting circles and reinforced both by ribs and by bars similar to the bar BB' (Figures 74 to 76). We determine the rigidities of, e. g., bars m_1m_2, m_2m_3, and m_3m_1 (Figure 76) from (5.50) and (5.53), setting $\varrho = R$, assigning to φ_0 the values $\frac{2\pi}{3}$, 0, $\frac{4\pi}{3}$ respectively.

§ 4. PURE BENDING OF A BOUNDED PLATE REINFORCED AROUND A HOLE

A bounded isotropic plate of arbitrary shape is subjected to uniformly distributed bending moments M along the outer edge (Figure 77). The hole in the plate has the shape shown in Figure 72. The solution of the first inverse problem is in this case given by (5.39), (5.50), and (5.53), with $\eta = 1$, i. e.,

$$a = c = -\frac{M}{2D(1+\nu)}. \tag{5.56}$$

The flexural rigidities (5.39) of the curved ribs of the equivalent reinforcing system are

$$A_i = R_i D(1+\nu) \qquad (i = 1, 2). \tag{5.57}$$

The flexural rigidity (5.50) of bar BB' is

$$A^* = \varrho D(1+\nu). \tag{5.58}$$

The torsional rigidities C_i and C^* remain arbitrary, since no stiffening rib or bar undergoes torsion. Hence, all the ribs BB_1B', BB_2B', and bar BB' have uniform cross sections. Let these cross sections be rectangular, their widths and heights being, respectively, h_1, b_1; h_2, b_2; h_*, b_*. We then obtain from (5.57) and (5.58) the following equivalence conditions for the reinforcing system:

$$b_* = \frac{\varrho}{1-\nu}\left(\frac{h}{h_*}\right)^3 \frac{E}{E_*}, \quad b_i = \frac{R_i}{1-\nu}\left(\frac{h}{h_i}\right)^3 \frac{E}{E_i} \qquad (i = 1, 2), \tag{5.59}$$

where h is the thickness and the modulus of elasticity of the plate; E_1, E_2, E_* are the moduli of elasticity of the stiffening ribs and bars, respectively.

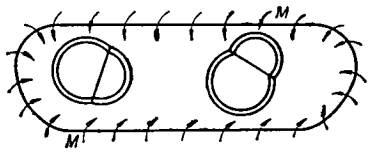


FIGURE 77.

Setting $E_1 = E_2 = E_*$ and $h_1 = h_2 = h_*$, we obtain from (5.59):

$$\frac{b_1}{R_1} = \frac{b_2}{R_2} = \frac{b_*}{\varrho} = \frac{1}{1-\nu}\left(\frac{h}{h_*}\right)^3 \frac{E}{E_*}. \tag{5.60}$$

We further require that plate and reinforcing system be of equal strength:

$$k_1^2\left|\frac{6M}{h^3}\right| = \frac{6}{b_1 h_1^2}|L_{r1}| = \frac{6}{b_2 h_2^2}|L_{r2}| = \frac{6}{b_* h_*^2}|M_u|, \tag{5.61}$$

where k_1^2 is the ratio of the permissible stresses for the materials of the rib and plate; $L_{ri}(i = 1, 2)$ and M_u are given by (5.37) and (5.44) for $a = c$.

Substituting these expressions in (5.61), we obtain

$$\frac{b_1}{R_1} = \frac{b_2}{R_2} = \frac{b_*}{\varrho} = \frac{h^2}{h_*^2}\cdot\frac{1}{k_1^2}. \tag{5.62}$$

If $\frac{b_1}{R_1} = \frac{b_2}{R_2} = \frac{b_*}{\varrho}$, we obtain from (5.60) and (5.62)

$$h_1 = h_2 = h_* = \frac{h}{k_1}\sqrt{\frac{\varrho}{b_*}}, \quad E_1 = E_2 = E_* = \frac{k_1^3 E}{1-\nu}\sqrt{\frac{b_*}{\varrho}}. \tag{5.63}$$

§ 5. TORSION OF A RECTANGULAR PLATE REINFORCED AROUND A HOLE

A rectangular isotropic plate is subjected to twisting moments $H_{xy} = H$ distributed uniformly along its edges. We shall solve the inverse problem of § 1 for a hole having the shape shown in Figure 72.

The deflection function is

$$w = bxy, \tag{5.64}$$

where

$$b = -\frac{H}{D(1-\nu)}. \tag{5.65}$$

We obtain from (5.21), (5.20), and (5.23)

$$\left.\begin{aligned} L_{ri} &= R_i H \sin 2\theta_i, \quad L_{\theta i} = R_i H \cos 2\theta_i \\ N_i &= -H\cos 2\theta_i \end{aligned}\right\}, \tag{5.66}$$

$$A_i = C_i = R_i D(1-\nu) \qquad (i = 1, 2). \tag{5.67}$$

The moments and shearing forces, which must act at B on the stiffening ribs BB_2B' and BB_1B', are

$$\left.\begin{aligned} L_{ri} &= -R_i H \sin 2(\varphi_0 \mp \varphi_i) \\ L_{\theta i} &= R_i H \cos 2(\varphi_0 \mp \varphi_i) \\ N_i &= -H\cos 2(\varphi_0 \mp \varphi_i) \end{aligned}\right\}. \tag{5.68}$$

The positive directions of the moments are shown in Figure 73. The force N_1 acts upward, and the force N_2 downward.

In addition, the plate acts on the rib at B with two concentrated forces caused by the twisting couple acting at the ends of the intersecting circular arcs,

$$P_i = \pm (H_{r\theta})_B = \pm H \cos 2(\varphi_i \mp \varphi_0) \quad (i = 1, 2). \tag{5.69}$$

The principal force vector, and the components of the moments (5.69) acting at point B are

$$P_z = 0, \quad M_u = -H\varrho \sin 2\varphi_0, \quad M_k = H\varrho \cos 2\varphi_0. \tag{5.70}$$

We similarly find for point B'

$$P_z' = 0, \quad M_u' = -M_u, \quad M_k' = -M_k. \tag{5.71}$$

We apply at B and B' the moments (5.70) and (5.71) by inserting between these points a straight bar possessing flexural rigidity A^* and torsional rigidity C^*. We determine these still unknown rigidities by requiring that strains and deflections of plate and bar be equal at B and B'. Proceeding similarly as in § 3, we find finally

$$C^* = A^* = \varrho D(1 - \nu). \tag{5.72}$$

If $\varphi_0 = 0$ or $\varphi_0 = \frac{\pi}{2}$, then $M_u = 0$ and the flexural rigidity is arbitrary, since the bar BB' undergoes no bending. Similarly, if $\varphi_0 = \frac{\pi}{4}$ then $M_k = 0$ and the torsional rigidity C^* is arbitrary.

All the results remain true when the hole has a more intricate shape, e. g., is bounded by several arcs of intersecting circles (cf., for instance, Figures 74 to 76).

§ 6. BENDING MOMENTS ACTING IN ONE SENSE ON A RECTANGULAR PLATE REINFORCED AROUND A HOLE

A rectangular isotropic plate is bent by applying uniformly distributed moments $M_y = M$. The hole is located anywhere in the plate and has the shape shown in Figure 78. The deflections of the plate are

$$w = a_1 x^2 + a_2 y^2, \tag{5.73}$$

where

$$a_1 = \frac{\nu M}{2D(1 - \nu^2)}, \quad a_2 = -\frac{M}{2D(1 - \nu^2)}. \tag{5.74}$$

In view of the arbitratiness established in § 1 of this chapter, and also for reasons of symmetry, we assume that the shearing force and the twisting moment acting on the stiffening rib vanish at B_0:

$$L_{0\tau} = N_0 = 0, \tag{5.75}$$

while the bending moment is

$$L_{0n} = \alpha l_1 M, \tag{5.76}$$

where α is a still unknown dimensionless coefficient, l_1 is the half-width of the hole. The factor l_1 has been introduced for convenience.

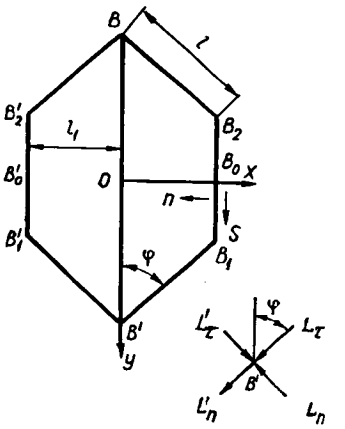


FIGURE 78.

We shall now consider the different parts of the stiffening system.

For part B_0B_1

$$\left.\begin{aligned} m(s) = p(s) = 0, \quad L_\tau = 0, \quad L_n = \alpha l_1 M \\ q = 0, \quad \frac{\partial^2 w}{\partial n \partial s} = 0, \quad \frac{\partial^2 w}{\partial s^2} = 2a_2 \end{aligned}\right\}. \tag{5.77}$$

Substituting (5.77) in (5.7), and taking (5.74) into account, we find

$$A = \alpha l_1 D(1 - \nu^2) \qquad C = \frac{0}{0}. \tag{5.78}$$

Hence, C is arbitrary. Part B_0B_1 undergoes only bending. The coefficient α must be positive.

For part B_1B'

$$m(s) = M \sin^2 \varphi, \quad p(s) = 0. \tag{5.79}$$

In addition, the plate acts on the rib at B_1 with the concentrated force $P = (H_{n\tau})_{B_1} = \frac{M}{2}\sin 2\varphi$ directed downward. Therefore

$$L_n = \frac{M}{2}(\alpha l - x_*)\sin 2\varphi,$$

$$L_\tau = M(\alpha l - x_*)\sin^2\varphi, \tag{5.80}$$

where x_* is the distance of the point considered from B_1.

For this part

$$\left.\begin{aligned} q = 0,\ \frac{\partial^2 w}{\partial s^2} &= 2(a_1\sin^2\varphi + a_2\cos^2\varphi) \\ \frac{\partial^2 w}{\partial n \partial s} &= (a_1 - a_2)\sin 2\varphi \end{aligned}\right\}. \tag{5.81}$$

Substituting (5.80) and (5.81) in (5.7), we find

$$\left.\begin{aligned} A &= (\alpha l - x_*)D(1-\nu^2)\frac{\sin\varphi\cos\varphi}{\cos^2\varphi - \nu\sin^2\varphi} \\ C &= (\alpha l - x_*)D(1-\nu)\operatorname{tg}\varphi \end{aligned}\right\}. \tag{5.82}$$

The rigidities A and C of part B_1B' vary linearly. A necessary and sufficient condition for these magnitudes to be positive and finite for all $0 \leqslant x_* \leqslant l$ is

$$\alpha \geqslant 1,\ \operatorname{tg}^2\varphi < \frac{1}{\nu}. \tag{5.83}$$

At point B' the plate acts on the reinforcing system with two concentrated forces, due to the twisting moments acting at the ends of parts B_1B' and $B_1'B'$. Each of these forces is equal to $P = (H_{n\tau})_{B'} = \frac{M}{2}\sin 2\varphi$; hence, the total concentrated force is $M\sin 2\varphi$, acting upward. We assume for parts $B_0'B_1'$ and $B_1'B'$ the same distribution of internal forces and moments, and the variation of the rigidities, as for parts B_0B_1 and B_1B'. We then find that the following moments act at B' both from the right and from the left:

$$L_\tau = -L_\tau' = Ml_1(\alpha - 1)\sin\varphi, \qquad L_n = L_n' = Ml_1(\alpha - 1)\cos\varphi. \tag{5.84}$$

The positive directions of these moments are shown in Figure 78. The x- and y-components of their sums are:

$$\left.\begin{aligned} M_u' &= (L_\tau' - L_\tau)\sin\varphi - (L_n + L_n')\cos\varphi \\ M_k' &= (L_\tau + L_\tau')\cos\varphi + (L_n' - L_n)\sin\varphi \end{aligned}\right\}. \tag{5.85}$$

Substituting (5.84) in (5.85), we find

$$M_u' = -2Ml_1(\alpha - 1),\quad M_k' = 0. \tag{5.86}$$

It is easily shown that the principal vector of the transverse forces acting at B' is zero. The condition of symmetry relative to the x-axis means

that a moment must act at B whose x-component is $M_u = -M_u'$. We therefore insert an elastic beam BB', subjected to pure bending by the moments M_u' and M_u' acting at its ends. We determine the flexural rigidity A_* of beam BB' by requiring that the following condition be satisfied at points B and B':

$$\frac{\partial w_*}{\partial y} = \frac{\partial w}{\partial y}. \tag{5.87}$$

This yields

$$A_* = \frac{M_u'}{2a_2} = 2l_1 D(1-\nu^2)(\alpha - 1). \tag{5.88}$$

The torsional rigidity of beam BB' is arbitrary. The coefficient $\alpha \geqslant 1$ should be selected in such a way that the rigidities of stiffening bar and ribs satisfy design and strength requirements.

Obviously, no beam BB' exists when $\alpha = 1$.

Since, according to (5.77), the plate transmits no loads to ribs B_1B_2 and $B_1'B_2'$, they need not be welded to the former.

The stiffening ribs, whose rigidities are given (5.78) and (5.82), are dimensioned in the same way as in § 8 of Chapter III. As an example we shall consider ribs of rectangular cross section $(b \times h_1)$. Table 22, which complements Table 14, gives values of ε_1 and ε_k [cf. (3.189)], and $\frac{C}{A} = \frac{\delta_2}{\delta_1}$ for two values of Poisson's ratio ν and different ratios $\left(\frac{h_1}{b}\right)$ of rib height to width.

TABLE 22

$\frac{h_1}{b}$	ε_1	ε_k	$\left(\frac{C}{A}\right)_{\nu=0.3}$	$\left(\frac{C}{A}\right)_{\nu=0.36}$
1.15	0.1267	0.1843	0.5593	0.5346
1.16	0.1301	0.1873	0.5538	0.5294
1.17	0.1335	0.1903	0.5484	0.5242
1.18	0.1369	0.1933	0.5431	0.5191
1.19	0.1404	0.1964	0.5378	0.5141
1.20	0.1440	0.1994	0.5326	0.5091
1.21	0.1476	0.2024	0.5274	0.5042
1.22	0.1513	0.2055	0.5223	0.4993
1.23	0.1551	0.2086	0.5173	0.4945
1.24	0.1589	0.2116	0.5123	0.4897
1.25	0.1628	0.2147	0.5074	0.4850
1.26	0.1667	0.2178	0.5025	0.4804
1.27	0.1707	0.2209	0.4978	0.4758
1.28	0.1748	0.2240	0.4930	0.4712
1.29	0.1789	0.2271	0.4883	0.4667
1.30	0.1831	0.2302	0.4836	0.4623

For ribs like B_1B' (Figure 78) we have by (5.82), for $\alpha = 1$ and $\varphi = 45°$:

$$A = (l - x_*)D(1+\nu),\ C = (l - x_*)D(1-\nu),\ \frac{C}{A} = \frac{1-\nu}{1+\nu}. \tag{5.89}$$

With $A = E_1 I = \frac{1}{12} E_1 h_1^3 b$, we obtain from the first equation (5.89) [for $E = E_1$]:

$$b = h \sqrt[4]{\left(\frac{b}{h_1}\right)^3 \cdot \frac{l}{h}\left(1 - \frac{x_*}{l}\right)\frac{1}{1-\nu}}. \tag{5.90}$$

In addition, for $\nu = 0.36$ we have $\frac{C}{A} = 0.4706$ by (5.89), and we find from Table 22, by linear interpolation:

$$\frac{h_1}{b} \cong 1.28. \tag{5.91}$$

Hence,

$$h_1 \cong 1.28h \sqrt[4]{\left(\frac{b}{h_1}\right)^3 \frac{l}{h}\left(1 - \frac{x_*}{l}\right)\frac{1}{1-\nu}}. \tag{5.92}$$

The dimensions h_1 and b of ribs like B_1B', calculated by (5.90) and (5.92) for $\nu = 0.36$, $h = 5$ mm, and $l = 29.2$ mm, are given in Table 23.

TABLE 23

$\frac{x_*}{l}$	b, mm	h_1, mm	$\frac{x_*}{l}$	b, mm	h_1, mm
0.0	7.48	9.59	0.6	5.95	7.62
0.1	7.29	9.34	0.7	5.54	7.09
0.2	7.07	9.07	0.8	5.00	6.41
0.3	6.84	8.77	0.9	4.21	5.39
0.4	6.58	8.44	0.95	3.54	4.53
0.5	6.29	8.06	1.0	0.00	0.00

The rigidity (5.78) of ribs B_1B_2 and $B_1'B_2'$ (of uniform cross section), subjected to bending only, is

$$A = \frac{1}{12} b h_1^3 E = E I_1 \frac{h^2}{12}. \tag{5.93}$$

Setting $h_1 = 9.55$ mm, we find $b = 3.32$ mm. The dimensions of the ribs reinforcing the edges of the hexagonal hole in the plate have thus been determined.

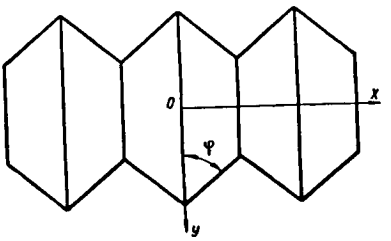


FIGURE 79.

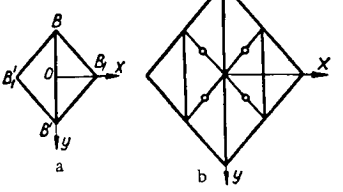


FIGURE 80.

If we cut a number of hexagonal holes in the plate (Figure 79) and reinforce the edges in the same way as before, the rigidities of the corresponding elements remain unchanged, except for the beams between two adjacent hexagons, whose flexural rigidities A_*' will be the sum of the flexural rigidities of ribs B_2B_1 and $B_2'B_1'$ (Figure 78):

$$A_*' = 2\alpha l_1 D(1-\nu^2). \tag{5.94}$$

Expressions (5.78), (5.82), and (5.88), remain valid under conditions (5.83) when the plate in the hole is rhombic or diamond-shaped (Figure 80). In this case the rigidities A_*', C_*' of the bars marked by small circles (Figure 80,b) are equal to the sum of the rigidities of ribs BB_2 and $B_1'B'$ in Figure 78:

$$\left.\begin{aligned} A_*' &= lD(1-\nu^2)(2\alpha-1)\frac{\sin\varphi\cos\varphi}{\cos^2\varphi-\nu\sin^2\varphi} \\ C_*' &= lD(1-\nu)(2\alpha-1)\operatorname{tg}\varphi \end{aligned}\right\}. \tag{5.95}$$

We can solve the first inverse problem in exactly the same way for holes of other shapes. When $\alpha = 1$, we obtain from (5.78), (5.82), and (5.88) the rigidities of ribs providing an equivalent reinforcement of the edge of a hole having the shape of a trapezium or a triangle (the right or left half of Figures 78 or 80,a).

Reinforcement of external recesses

The above solution applies also to the equivalent reinforcement of external recesses of trapezoidal or triangular shape (Figure 81,a) in an isotropic plate subjected to bending moments about one axis. When $\alpha = 1$, no beam BB' exists. The rigidities of ribs B_1B, B_1B', $B_1'B$, $B_1'B'$, and B_1B_2 are given, respectively, by (5.82) and (5.78) for $\alpha = 1$.

§ 7. DIFFERENT FORM OF THE SOLUTION OF THE FIRST INVERSE PROBLEM FOR AN ISOTROPIC PLATE

A thin isotropic plate carries an arbitrary load $q(x, y)$. We assume that a region S of the middle plane xy of the plate is free from loads. We cut out this part, and reinforce the edge Γ of the hole thus formed by an elastic rib system equivalent to the removed part of the plate (§ 1 of this chapter). For this we must solve the first inverse problem and determine the rigidities of the members of the reinforcing system.

We assume that the problem of bending of the solid plate has been solved. The state of stress in region S is described by two supposed to be known holomorphic functions, $\varphi(z)$ and $\psi(z)$, of the complex variable $z = x + iy$.

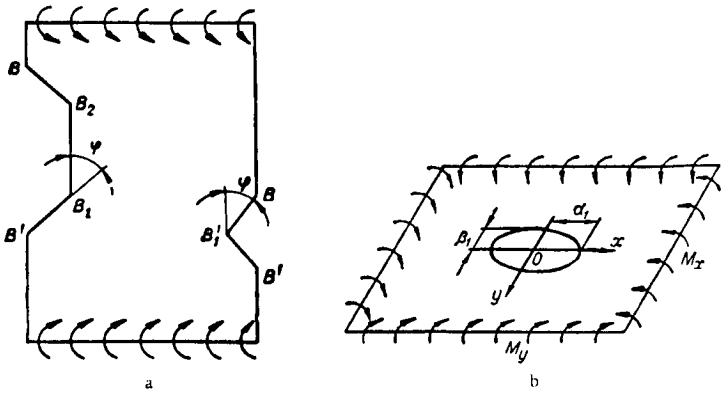


FIGURE 81.

The boundary condition along the joint Γ between the plate and the curved stiffening rib is (cf. (3.2) and (3.6)):

$$-\frac{d}{ds}N_2(t,\bar{t}) = \frac{D}{4i}\left[N_1(t,\bar{t})\left(\frac{1}{C}+\frac{1}{A}\right)+\dot{t}^2\overline{N_1(t,\bar{t})}\left(\frac{1}{C}-\frac{1}{A}\right)\right], \tag{5.96}$$

where

$$\left.\begin{aligned} N_1(t,\bar{t}) &= 2D(1-\nu)[\varkappa\overline{\varphi(t)} - \bar{t}\varphi'(t) - \psi(t)] - iC_1^* t + C_2^* \\ N_2(t,\bar{t}) &= D[\overline{\varphi(t)} + \bar{t}\varphi'(t) + \psi(t)] \end{aligned}\right\}, \tag{5.97}$$

t is the affix of the point on Γ, $\dot{t} = \frac{dx}{ds} + i\frac{dy}{ds} = \frac{dt}{ds}$, C_1^*, C_2^* are, respectively, real and complex arbitrary constants, $\varkappa = (3+\nu)/(1-\nu)$. The counterclockwise direction is taken as positive on Γ.

The functions N_1 and N_2 correspond to the basic deflection $w^0(x,y)$ of the solid plate and are considered as known. From (5.96) and the conjugate equation, we obtain

$$\left.\begin{aligned} C &= -\frac{D\,\mathrm{Re}\,(iN_1^2\dot{t}^2)}{2\,\mathrm{Re}\left[N_1\left(\frac{d}{ds}\overline{N}_2 - \dot{t}^2\frac{d}{ds}N_2\right)\right]} \\ A &= \frac{D\,\mathrm{Re}\,(iN_1^2\dot{t}^2)}{2\,\mathrm{Re}\left[N_1\left(\frac{d}{ds}\overline{N}_2 + \dot{t}^2\frac{d}{ds}N_2\right)\right]} \end{aligned}\right\}. \tag{5.98}$$

As example we consider bending of a rectangular plate by applying moments $M_x = \text{const.}$ and $M_y = \eta M_x = \text{const.}$ at the edges. The plate has an elliptic hole (Figure 81,b) whose contour is given in the form

$$\frac{x^2}{\alpha_1^2} + \frac{y^2}{\beta_1^2} = 1,$$

whence

$$\left.\begin{aligned} \frac{dy}{dx} &= -\frac{\beta_1^2}{\alpha_1^2}\cdot\frac{x}{y},\quad \left(\frac{dx}{ds}\right)^2 = \dot{x}^2 = \frac{1}{1+\left(\frac{dy}{dx}\right)^2} = \frac{\alpha_1^4 y^2}{\alpha_1^4 y^2 + \beta_1^4 x^2} \\ \dot{y}^2 &= \left(\frac{dy}{dx}\dot{x}\right)^2 = \frac{\beta_1^4 x^2}{\alpha_1^4 y^2 + \beta_1^4 x^2},\quad \dot{t} = \dot{x} + i\dot{y} = \frac{y - ikx}{\sqrt{y^2 + k^2 x^2}} \end{aligned}\right\}. \tag{5.99}$$

When the same bending moments are applied to a solid plate, we have

$$\varphi(z) = -\frac{M_x(1+\eta)}{8D(1+\nu)}z,\quad \psi(z) = \frac{M_x(\eta-1)}{4D(1-\nu)}z. \tag{5.100}$$

TABLE 24

$\frac{y}{\beta_1}$	$\frac{A}{\beta_1 D}$	$\frac{C}{\beta_1 D}$	$\frac{b}{h}\sqrt[4]{\frac{E_1}{E}\cdot\frac{h}{\beta_1}}$	$\frac{h_1}{h}\sqrt[4]{\frac{E_1}{E}\cdot\frac{h}{\beta_1}}$
0	1.608	1.485	1.601	1.032
0.2	1.585	1.470	1.605	1.027
0.4	1.516	1.424	1.614	1.011
0.6	1.410	1.345	1.616	0.986
0.8	1.265	1.224	1.604	0.953
1.0	1.071	1.050	1.566	0.909

Substituting (5.100) in (5.97), and then (5.97) and (5.99) in (5.98), setting $C_1^* = C_2^* = 0$, we obtain the rigidities of the stiffening rib:

$$\left.\begin{aligned} C &= D(1-\nu)\sqrt{y^2+k^2x^2}\ \frac{\eta-k}{k(\eta-1)}, \\ A &= \frac{D(1-\nu^2)\sqrt{y^2+k^2x^2}(kx^2\eta+y^2)}{k^2x^2(\eta-\nu)-y^2(\nu\eta-1)}, \end{aligned}\right\} \tag{5.101}$$

where $k = \beta_1^2/\alpha_1^2$.

An equivalent rib exists if A and C are positive and finite. This occurs in the following cases:

1) $1 < \eta \leqslant \nu^{-1}$ and $k \leqslant \eta$;
2) $\nu < \eta < 1$ and $k \geqslant \eta$;
3) $\eta \leqslant 0$ and $k = \eta(1-\nu\eta)/(\eta-\nu)$;
4) $\eta = 1$ and $k = 1$.

Table 24 gives the values of A,C and of the corresponding cross-sectional dimensions of the stiffening rib (h is the thickness of the plate; E_1, E are Young's moduli for rib and plate, respectively; h_1, b are height and width of the rectangular rib cross section), calculated for $\eta = 0.5$, $k = 2$, and $\nu = 0.3$.

In the particular case of a circular hole ($k = 1$), we obtain from (5.101) the same values of the rigidities as from (5.27):

$$C = RD(1-\nu),\quad A = \frac{RD(1-\nu^2)(x^2\eta + y^2)}{x^2(\eta-\nu) - y^2(\nu\eta - 1)}. \tag{5.102}$$

§ 8. SECOND INVERSE PROBLEM

We shall call neutral a hole with reinforced edge in a thin isotropic plate, which does not alter the basic deflections $w^0(x, y)$ of the plate. Obviously, no stress concentrations arise near such a hole.

The second inverse problem consists in determining the shape of a neutral hole with reinforced edge Γ, when the load on the plate and the rigidity of the stiffening rib are given.

We solve this problem by using (5.96), rewritten as follows:

$$a_1\dot{t}^3 + a_2\dot{\bar{t}} + a_3\dot{t} + a_4 = 0, \tag{5.103}$$

where $\dot{t} = \frac{dt}{ds}$ has to be determined,

$$a_1 = D\left(\frac{1}{C} - \frac{1}{A}\right)N_1(t, \bar{t}),\quad a_2 = -4i\frac{\partial}{\partial \bar{t}}\overline{N}_2(t, \bar{t}),$$
$$a_4 = D\left(\frac{1}{C} + \frac{1}{A}\right)\overline{N}_1(t, \bar{t}),\quad a_3 = -4i\frac{\partial}{\partial t}\overline{N}_2(t, \bar{t}). \tag{5.104}$$

The functions a_1, a_2, a_3, a_4 are assumed to be known. The complex conjugate equation to (5.103) is

$$\bar{a}_1\dot{\bar{t}}^3 + \bar{a}_2\dot{t} + \bar{a}_3\dot{\bar{t}} + \bar{a}_4 = 0. \tag{5.105}$$

The system of nonlinear equations (5.103), (5.105) serves for determining the two unknown functions $\dot{t}$ and $\dot{\bar{t}}$. Furthermore

$$t\dot{\bar{t}} = 1. \tag{5.106}$$

Hence, the problem cannot be solved when a_1 and a_4 are chosen arbitrarily. If the basic stresses in the plate, i. e., the functions $N_1(t, \bar{t})$ and $N_2(t, \bar{t})$, are known, only one of the two functions $A(s)$ and $C(s)$ can be specified. We shall assume that only the ratio $\frac{A}{C} = \lambda$ is given. To determine the unknown functions $\dot{t}$ and $\dot{\bar{t}}$, we multiply (5.103) by $\dot{\bar{t}}$; by (5.106) we obtain

$$a_1\dot{t}^2 + a_2\dot{t} + a_4\dot{\bar{t}} + a_3 = 0. \tag{5.107}$$

Multiplying (5.105) by a_2 and (5.107) by $\bar{a}_1$, and subtracting the second equation thus obtained from the first, we obtain

$$(\bar{a}_2a_3 - \bar{a}_1a_4)\dot{\bar{t}} + (\bar{a}_2a_2 - \bar{a}_1a_1)\dot{t} + (\bar{a}_4a_2 - \bar{a}_1a_3) = 0. \tag{5.108}$$

The corresponding complex conjugate equation is

$$(\bar{a}_3a_2 - \bar{a}_1a_1)\dot{\bar{t}} + (\bar{a}_2a_3 - \bar{a}_4a_1)\dot{t} + (\bar{a}_2a_4 - \bar{a}_3a_1) = 0. \tag{5.109}$$

From (5.108) and (5.109), we find

$$\dot{t} = \frac{(\bar{a}_2a_4 - \bar{a}_3a_1)(\bar{a}_3a_2 - \bar{a}_1a_4) - (\bar{a}_4a_2 - \bar{a}_1a_3)(\bar{a}_2a_3 - \bar{a}_1a_1)}{(\bar{a}_2a_2 - \bar{a}_1a_1)^2 - (\bar{a}_2a_3 - \bar{a}_4a_1)(\bar{a}_3a_2 - \bar{a}_1a_4)}. \tag{5.110}$$

Substituting (5.110) in (5.106), we obtain an algebraic equation of the fourth degree in A^2:

$$b_0 + b_1A^2 + b_2A^4 + b_3A^6 + b_4A^8 = 0, \tag{5.111}$$

where the functions $b_k (k = 0, 1, 2, 3, 4)$ are expressed through the given functions $N_1(t, \bar{t})$, $N_2(t, \bar{t})$ and λ.

Solving (5.111), we find A as a function of x, y. This function must be positive.

We determine the contour equation of the neutral hole from the identity

$$\frac{dy}{dx} = \frac{i(\dot{\bar{t}} - \dot{t})}{\dot{t} + \dot{\bar{t}}}. \tag{5.112}$$

Substituting for $\dot{t}$ from (5.110), we obtain

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}, \tag{5.113}$$

where

$$Q(x, y) = \operatorname{Re}[(\bar{a}_2a_4 - \bar{a}_3a_1)(\bar{a}_3a_2 - \bar{a}_1a_4 - \bar{a}_4a_2 + \bar{a}_1a_1)],$$

$$P(x, y) = \operatorname{Im}[(\bar{a}_2a_4 - \bar{a}_3a_1)(\bar{a}_3a_2 - \bar{a}_1a_4 + \bar{a}_2a_2 - \bar{a}_1a_1)],$$

are known real functions of x, y.

The one-parameter family of integral curves of (5.113) is the solution of the second inverse problem. By (5.111) each integral curve corresponds to a rigidity A.

This inverse problem does not always have a solution, since the contour of the neutral hole must be closed. Certain sufficient criteria for the existence of closed integral curves of (5.113) are given in /86, 110, 133/.

The problem is considerably simplified if we set $\frac{A}{C} = \lambda = 1$.

In this case $a_1 = 0$, while (5.110) and (5.111) become

$$\dot{t} = \frac{a_4\bar{a}_2 - a_3\bar{a}_4}{a_2\bar{a}_2 - a_3\bar{a}_3}, \tag{5.114}$$

$$b_0 + b_2A^4 = 0, \tag{5.115}$$

where

$$b_0 = D^2\left(\overline{N}_1\frac{\partial N_2}{\partial \bar{t}} + N_1\frac{\partial \overline{N}_2}{\partial \bar{t}}\right)\times\left(N_1\frac{\partial \overline{N}_2}{\partial t} + \overline{N}_1\frac{\partial N_2}{\partial t}\right),$$

$$b_1 = -4\left(\frac{\partial \overline{N}_2}{\partial \bar{t}}\cdot\frac{\partial N_2}{\partial t} - \frac{\partial \overline{N}_2}{\partial t}\cdot\frac{\partial N_2}{\partial \bar{t}}\right)^2. \tag{5.116}$$

Solving (5.115), we find

$$C = A = -\frac{D\left|\overline{N}_1\frac{\partial}{\partial \bar{t}}N_2 + N_1\frac{\partial}{\partial \bar{t}}\overline{N}_2\right|}{2\left|\frac{\partial \overline{N}_2}{\partial t}\cdot\frac{\partial N_2}{\partial t} - \frac{\partial \overline{N}_2}{\partial t}\cdot\frac{\partial N_2}{\partial \bar{t}}\right|}. \tag{5.117}$$

Equation (5.113) assumes the form

$$\frac{dy}{dx} = \frac{\operatorname{Re} N_3}{\operatorname{Im} N_3}, \tag{5.118}$$

where

$$N_3 = \overline{N}_1\frac{\partial N_2}{\partial t} + N_1\frac{\partial \overline{N}_2}{\partial t}.$$

In this case the shape of the neutral hole is independent of the rigidity A.

We shall consider some examples.

a) Twisting of a bounded rectangular plate by moments $H_{xy} = H = \text{const}$. In this case

$$N_1 = -2iHt, \quad N_2 = \frac{iHt}{1-\nu}. \tag{5.119}$$

Substituting (5.119) in (5.118), we find

$$\frac{dy}{dx} = -\frac{x}{y}. \tag{5.120}$$

Integration yields the contour equation of the neutral hole:

$$x^2 + y^2 = R^2, \tag{5.121}$$

where R^2 is an arbitrary integration constant. The rigidities of the thin rib reinforcing the edge of this circular hole are, according to (5.117):

$$C = A = RD(1-\nu). \tag{5.122}$$

This coincides with the result obtained in § 2 by a different procedure (cf. (5.24)).

b) Bending of a bounded rectangular plate by moments $M_x = M =$ const. and $M_y = \eta M$ applied at the edges. In this case

$$\left.\begin{aligned} N_1 &= -\frac{M}{2}[(\eta+1)\bar{t}+(\eta-1)t] \\ N_2 &= \frac{M}{4}\left[\frac{\eta-1}{1-\nu}t - \frac{\eta+1}{1+\nu}\bar{t}\right] \end{aligned}\right\}. \tag{5.123}$$

We obtain from (5.118)

$$\frac{dy}{dx} = -\frac{\eta(1-\nu\eta)}{\eta-\nu}\cdot\frac{x}{y}. \tag{5.124}$$

Integration of this equation yields the contour equation of the neutral hole

$$y^2 + \frac{\eta(1-\nu\eta)}{\eta-\nu}x^2 = \beta^2, \tag{5.125}$$

where β^2 is an integration constant. When $\nu < \eta < \frac{1}{\nu}$ and also when $\eta < 0$, (5.125) represents an ellipse.

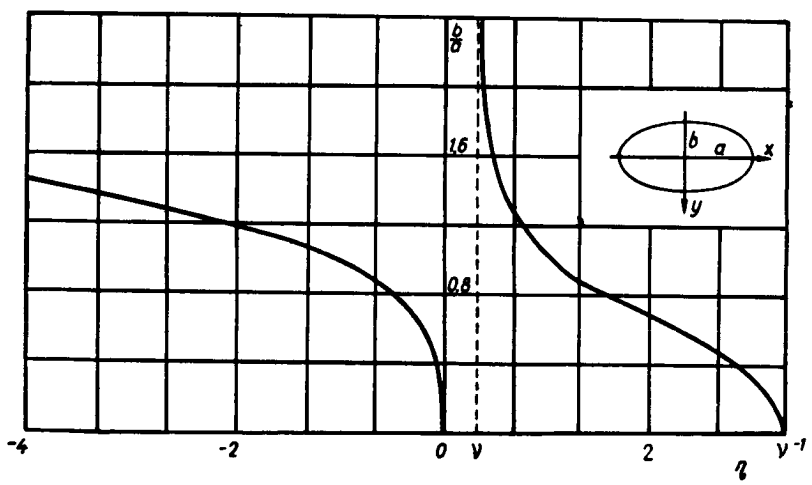


FIGURE 82.

The rigidities of the stiffening rib are by (5.117)

$$C = A = D(1-\nu^2)\frac{\sqrt{k^2x^2+y^2}}{|1-\nu\eta|}, \tag{5.126}$$

where $k = \eta(1-\nu\eta)/(\eta-\nu)$ is the ratio of the squares of the ellipse semiaxes. Figure 82 shows the ratio $\frac{b}{a}$ of the semiaxes of the neutral elliptic holes as function of $\eta = \frac{M_y}{M_x}$ and $\nu = 0.3$.

§ 9. INVERSE UNIFIED CONTACT PROBLEM WITH APPROXIMATE BOUNDARY CONDITIONS

By analogy with §§ 7 and 8 we formulate the problem of determining the shape of a neutral hole in a plate, and the corresponding rigidity of an equivalent stiffening rib when approximate boundary conditions are satisfied on Γ.

We assume that some part S, which carries no external load, is cut from the plate, and that the edge of this cut-out is reinforced by a very thin equivalent rib (cf. § 1 of Chapter IV). The stresses in the solid plate are considered as known. In domain S this state of stress is defined by two holomorphic functions $\varphi(z)$ and $\psi(z)$, also considered as known.

The boundary condition along the joint Γ between the plate and the thin stiffening rib is given by (4.23). In the case considered the right-hand side vanishes, while the functions $t(s)$ and $K(s)$, which, respectively, define the shape of the neutral hole and the rigidity of the equivalent stiffening rib, are unknown. Separating the real and imaginary parts in (4.23), we obtain

$$\operatorname{Re}\Phi_1(t,\ \bar{t}) = -\dot{y}K\operatorname{Re}\Phi_2(t,\ \bar{t}) \tag{5.127}$$

$$\operatorname{Im}\Phi_1(t,\ \bar{t}) = -\dot{x}K\operatorname{Re}\Phi_2(t,\ \bar{t}) \tag{5.128}$$

where

$$\left.\begin{aligned} \Phi_1(t,\ \bar{t}) &= a_1\overline{\varphi(t)} + \bar{t}\varphi'(t) + \psi(t) \\ \Phi_2(t,\ \bar{t}) &= i\frac{d}{ds}[a_2\overline{\varphi(t)} - \bar{t}\varphi'(t) - \psi(t)] \\ \dot{x} &= \frac{dx}{ds},\quad \dot{y} = \frac{dy}{ds} \end{aligned}\right\}. \tag{5.129}$$

Dividing (5.127) by (5.128), we obtain

$$y' = \frac{dy}{dx} = \frac{\operatorname{Re}\Phi_1(t,\ \bar{t})}{\operatorname{Im}\Phi_1(t,\ \bar{t})}. \tag{5.130}$$

Equation (5.130) defines some family of curves Γ. If these curves are closed, Γ will represent the contour of the neutral hole (cf. § 8). The clockwise direction is here considered as the positive sense of describing Γ (cf. Figures 45 and 46).

We have

$$i\frac{d}{ds} = (\dot{x} + i\dot{y})\left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}\right) = \dot{x}^2(1 + iy')\left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right). \tag{5.131}$$

Substituting (5.129) and (5.131) in (5.128), we find the rigidity of the equivalent rib:

$$K = \frac{-\operatorname{Im}[a_1\overline{\varphi(t)} + \bar{t}\varphi(t) + \psi(t)]}{\dot{x}^3\operatorname{Re}\left\{(1 + iy')\left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)[a_2\overline{\varphi(t)} - \bar{t}\varphi'(t) - \psi(t)]\right\}}. \tag{5.132}$$

where

$$\dot{x} = \pm \sqrt{\frac{1}{1 + y'^2}} \tag{5.133}$$

and y' is given by (5.130). The plus sign in (5.133) corresponds to the two-dimensional problem, and the minus sign, to bending like a thin plate. These two cases will now be considered separately.

Generalized plane state of stress

When $a_1 = 1$, the function $\Phi_1(t, \bar{t})$ in (5.129) is expressed (/82/, p. 112) through Airy's stress function $\Phi(x, y) = \Phi$:

$$\Phi_1(t, \bar{t}) = \frac{\partial \Phi}{\partial x} - i \frac{\partial \Phi}{\partial y}. \tag{5.134}$$

Hence, by (5.130),

$$\frac{dy}{dx} = -\frac{\dfrac{\partial \Phi}{\partial x}}{\dfrac{\partial \Phi}{\partial y}}. \tag{5.135}$$

Integrating (5.135), we obtain the contour equation of the neutral hole

$$\Phi + C = 0, \tag{5.136}$$

where C is a real constant.

Since the stress function is determined up to linear terms in x and y, we can write (5.136) in the form

$$\Phi + ax + by + c = 0, \tag{5.137}$$

where a and b are real constants.

The same equation was obtained by a different procedure in /209/, which contains several examples of the equivalent reinforcement of hole edges in plates undergoing tension or pure shearing.

Bending like thin plates

For bending like thin plates, i. e., for $a_1 = -\dfrac{3+\nu}{1-\nu}$, we obtain by comparing (5.129) with (1.77)

$$\Phi_1(t, \bar{t}) = a_1 \overline{\varphi(t)} + \bar{t}\varphi'(t) + \psi(t) =$$
$$= -\frac{4}{1-\nu} \overline{\varphi(t)} + \frac{1}{2} \left(\frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right)_\Gamma, \tag{5.138}$$

where w is the deflection of the plate.

Substituting (5.138) in (5.130), we obtain

$$\frac{dy}{dx} = \frac{(1-\nu)\frac{\partial w}{\partial x} - 8\operatorname{Re}\varphi(t)}{-(1-\nu)\frac{\partial w}{\partial y} + 8\operatorname{Im}\varphi(t)}, \tag{5.139}$$

whence

$$(1-\nu)\left(\frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy\right) - 8\operatorname{Re}[\varphi(t)\,d\bar{t}] = 0, \tag{5.140}$$

or, after integration,

$$(1-\nu)w - 8\int \operatorname{Re}[\varphi(t)\,d\bar{t}] = c, \tag{5.141}$$

where c is a real constant.

If we represent the function $\varphi(z)$ in the form

$$\varphi(z) = p(x, y) + iq(x, y), \tag{5.142}$$

we can write (5.141) as follows:

$$(1-\nu)w - 8\int [p(x\ \ y)\,dx + q(x, y)\,dy] = c. \tag{5.143}$$

This is the required equation of the contours Γ of the neutral holes.

As example we shall consider the problem of bending of a rectangular thin bounded plate subjected to moments M_x and $M_y = \eta M_x$.

We have for the solid plate

$$w = \frac{1}{2D(1-\nu^2)}[(\nu M_y - M_x)x^2 - (M_y - \nu M_x)y^2]. \tag{5.144}$$

In accordance with (4.10), we have for $w^0 = 0$:

$$\varphi(z) = -\frac{M_x + M_y}{8D(1+\nu)}z, \qquad \psi(z) = \frac{M_y - M_x}{4D(1-\nu)}z, \tag{5.145}$$

whence

$$p(x, y) = -\frac{M_x + M_y}{8D(1+\nu)}x, \qquad q(x, y) = -\frac{M_x + M_y}{8D(1-\nu)}y. \tag{5.146}$$

Substituting (5.144) and (5.146) in (5.143), we obtain the equation of Γ:

$$\eta x^2 + y^2 = C. \tag{5.147}$$

For $\eta > 0$ this represents the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{\eta a^2} = 1, \tag{5.148}$$

where a is an arbitrary real constant equal to the semiaxis lying in the x-direction.

TABLE 25

$\frac{x}{a}$	$1/\sqrt{\eta}$								
	0.3	0.5	0.7	1.0	1.2	1.5	2.0	2.5	2.8
	A/aD								
0	∞	3.218	1.904	1.3	1.107	0.9289	0.7568	0.6540	0.6089
0.1	19.00	3.128	1.899	1.3	1.107	0.9284	0.7552	0.6522	0.6067
0.2	6.837	2.933	1.886	1.3	1.105	0.9223	0.7497	0.6457	0.6010
0.3	4.819	2.749	1.870	1.3	1.102	0.9170	0.7406	0.6359	0.5906
0.4	4.350	2.633	1.857	1.3	1.098	0.9081	0.7277	0.6215	0.5761
0.5	4.358	2.591	1.850	1.3	1.095	0.8983	0.7117	0.6034	0.5572
0.6	4.563	2.609	1.853	1.3	1.091	0.8859	0.6918	0.5809	0.5338
0.7	4.869	2.669	1.866	1.3	1.085	0.8742	0.6705	0.5543	0.5060
0.8	5.242	2.767	1.891	1.3	1.081	0.8650	0.6524	0.5285	0.4805
0.9	5.651	2.890	1.924	1.3	1.079	0.8659	0.6525	0.5208	0.4642
1.0	6.087	3.027	1.966	1.3	1.082	0.9007	0.8045	0.9208	1.215

Substituting (5.145) and (5.147) in (5.132), we obtain the variable flexural rigidity of a thin rib reinforcing the edge of the elliptic hole:

$$A = \frac{aD(1-\nu^2)\sqrt{\eta}f^3}{f-\nu}, \tag{5.149}$$

where

$$f = \frac{1}{\sqrt{\eta}}\left[\frac{x^2}{a^2}(\eta-1)+1\right]. \tag{5.150}$$

Obviously, A is finite for $f > \nu$, i.e., for $\nu^2 < \eta < \frac{1}{\nu^2}$.

Values of the flexural rigidity $\frac{A}{D}$ ((5.149)) are given in Table 25 for some values of $\eta = M_y/M_x$. We see that in the range $0.60 \leqslant \eta \leqslant 1.7$ the rigidity is nearly constant (to within 5%).

For $0 < \eta \leqslant \nu^2$ the flexural rigidity A becomes infinite at points whose abscissae are

$$x_\infty = \pm a\sqrt{\frac{\nu\sqrt{\eta}-1}{\eta-1}}.$$

Chapter VI

OTHER PROBLEMS OF PLATES WITH STIFFENING RIBS

§ 1. BENDING OF PLATES SHAPED LIKE ANNULAR SECTORS

Consider an arbitrarily loaded thin plate bounded by two arcs of concentric circles and two radii (Figure 83). The curved edges of the plate are reinforced by thin elastic ribs of uniform cross section. The plate is simply supported at all edges.

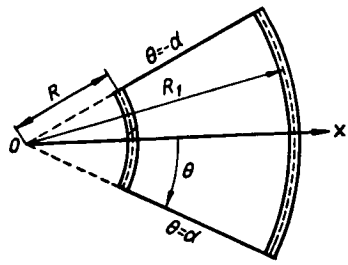


FIGURE 83.

We shall use a polar system of coordinates (r, θ) with origin at the center of curvature of the arcs. The polar axis coincides with the axis of symmetry Ox of the plate. The equation of the elastic surface of the plate is then (cf. /20/, pp. 209, 321):

$$w = w^0(r, \theta) + w^1(r, \theta) = w^0(r, \theta) + \sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{2\alpha}} \sin\frac{n\pi(\alpha+\theta)}{2\alpha} +$$

$$+ \sum_{n=1}^{\infty} B_n r^{-\frac{n\pi}{2\alpha}} \sin\frac{n\pi(\alpha+\theta)}{2\alpha} + \sum_{n=1}^{\infty} C_n r^{\frac{n\pi}{2\alpha}+2} \sin\frac{n\pi(\alpha+\theta)}{2\alpha} +$$

$$+ \sum_{n=1}^{\infty} D_n r^{-\frac{n\pi}{2\alpha}+2} \sin\frac{n\pi(\alpha+\theta)}{2\alpha}, \qquad (6.1)$$

where $w^0(r, \theta)$ are the basic deflections, considered as known, of the non-reinforced plate under the same load; $w^1(r, \theta)$ are the additional deflections

due to the stiffening ribs. Expressions for $w^0(r, \theta)$ were given by Galerkin /20/ for different kinds of load.

The unknown coefficients A_n, B_n, C_n, D_n must be determined from the following boundary conditions:

at $\theta = \pm\alpha$

$$w = 0, \qquad \frac{\partial^2 w}{\partial \theta^2} = 0; \tag{6.2}$$

at $r = R_1$, according to (2.364)

$$(\delta_1 + \nu)\frac{\partial w}{\partial r} + R_1\frac{\partial^2 w}{\partial r^2} - \delta_2\frac{\partial^2 w}{\partial r \partial \theta^2} = 0, \; w = 0, \tag{6.3}$$

at $r = R$, according to (2.365)

$$(\delta_1' - \nu)\frac{\partial w}{\partial r} - R\frac{\partial^2 w}{\partial r^2} - \delta_2'\frac{\partial^2 w}{\partial r \partial \theta^2} = 0, \; w = 0, \tag{6.4}$$

where

$$\delta_1 = \frac{A}{R_1 D}, \quad \delta_2 = \frac{C}{R_1 D}, \quad \delta_1' = \frac{A'}{RD}, \quad \delta_2' = \frac{C'}{RD}.$$

The function $w^0(r, \theta)$ satisfies the conditions

$$w^0 = 0, \; \frac{\partial^2 w^0}{\partial \theta^2} = 0 \text{ at } \theta = \pm\alpha, \tag{6.5}$$

$$w^0 = 0, \; \frac{\partial^2 w^0}{\partial r^2} + \frac{\nu}{r}\cdot\frac{\partial w^0}{\partial r} = 0 \text{ at } r = R \text{ and } r = R_1. \tag{6.6}$$

Substituting (6.1) in (6.2) to (6.4), and taking (6.5) and (6.6) into account, we obtain the following boundary conditions for the additional deflections w^1:

$$w^1 = 0, \; (\delta_1 + \nu)\frac{\partial w^1}{\partial r} + R_1\frac{\partial^2 w^1}{\partial r^2} - \delta_2\frac{\partial^2 w^1}{\partial r \partial \theta^2} = f_1(\theta) \text{ at } r = R_1, \tag{6.7}$$

$$w^1 = 0, \; -(\delta_1' - \nu)\frac{\partial w^1}{\partial r} + R\frac{\partial^2 w^1}{\partial r^2} + \delta_2'\frac{\partial^2 w^1}{\partial r \partial \theta^2} = f_2(\theta) \text{ at } r = R, \tag{6.8}$$

where

$$\left.\begin{aligned} f_1(\theta) &= \left[-\delta_1\frac{\partial w^0}{\partial r} + \delta_2\frac{\partial^2 w^0}{\partial r \partial \theta^2}\right]_{r=R_1} \\ f_2(\theta) &= \left[\delta_1'\frac{\partial w^0}{\partial r} - \delta_2'\frac{\partial^2 w^0}{\partial r \partial \theta^2}\right]_{r=R} \end{aligned}\right\}. \tag{6.9}$$

The conditions at $\theta = \pm\alpha$ are satisfied automatically.

We expand the known functions $\left(\frac{\partial w^0}{\partial r}\right)_{r=R}$ and $\left(\frac{\partial w^0}{\partial r}\right)_{r=R_1}$ in trigonometric series for $-\alpha<\theta<+\alpha$:

$$\left.\begin{aligned}\left(\frac{\partial w^0}{\partial r}\right)_{r=R}&=\sum_{n=1}^{\infty}\mu_n\sin\frac{n\pi(\alpha+\theta)}{2\alpha},\\ \left(\frac{\partial w^0}{\partial r}\right)_{r=R_1}&=\sum_{n=1}^{\infty}\lambda_n\sin\frac{n\pi(\alpha+\theta)}{2\alpha},\end{aligned}\right\}\tag{6.10}$$

where λ_n and μ_n are found from Fourier's formulas:

$$\left.\begin{aligned}\lambda_n&=\frac{1}{\alpha}\int_{-\alpha}^{+\alpha}\left(\frac{\partial w^0}{\partial r}\right)_{r=R_1}\sin\frac{n\pi(\alpha+\theta)}{2\alpha}\,d\theta\\ \mu_n&=\frac{1}{\alpha}\int_{-\alpha}^{+\alpha}\left(\frac{\partial w^0}{\partial r}\right)_{r=R}\sin\frac{n\pi(\alpha+\theta)}{2\alpha}\,d\theta\end{aligned}\right\}.\tag{6.11}$$

Hence,

$$\left.\begin{aligned}\left(\frac{\partial^3 w^0}{\partial r\partial\theta^2}\right)_{r=R}&=-\sum_{n=1}^{\infty}\mu_n\frac{n^2\pi^2}{4\alpha^2}\sin\frac{n\pi(\alpha+\theta)}{2\alpha}\\ \left(\frac{\partial^3 w^0}{\partial r\partial\theta^2}\right)_{r=R_1}&=-\sum_{n=1}^{\infty}\lambda_n\frac{n^2\pi^2}{4\alpha^2}\sin\frac{n\pi(\alpha+\theta)}{2\alpha}\end{aligned}\right\}.\tag{6.12}$$

Using (6.12) and (6.10), (6.9) can be rewritten thus:

$$\left.\begin{aligned}f_1(\theta)&=-\sum_{n=1}^{\infty}\left(\delta_2\frac{n^2\pi^2}{4\alpha^2}+\delta_1\right)\lambda_n\sin\frac{n\pi(\alpha+\theta)}{2\alpha}\\ f_2(\theta)&=\sum_{n=1}^{\infty}\left(\frac{n^2\pi^2}{4\alpha^2}\delta_2'+\delta_1'\right)\mu_n\sin\frac{n\pi(\alpha+\theta)}{2\alpha}\end{aligned}\right\}.\tag{6.13}$$

Substituting for w^1 from (6.1) in (6.7) and (6.8), and taking (6.13) into account, we obtain the following four equations for determining the coefficients A_n, B_n, C_n, D_n:

$$\left.\begin{aligned}&A_nR_1^{\frac{n\pi}{2\alpha}}+B_nR_1^{-\frac{n\pi}{2\alpha}}+C_nR_1^{\frac{n\pi}{2\alpha}+2}+D_nR^{-\frac{n\pi}{2\alpha}+2}=0\\ &A_nR^{\frac{n\pi}{2\alpha}}+B_nR^{-\frac{n\pi}{2\alpha}}+C_nR^{\frac{n\pi}{2\alpha}+2}+D_nR^{-\frac{n\pi}{2\alpha}+2}=0\\ &\frac{n\pi}{4\alpha^2}\left[n\pi-2\alpha(1-\nu-\delta_1)+\frac{n^2\pi^2}{2\alpha}\delta_2\right]A_nR_1^{\frac{n\pi}{2\alpha}}+\end{aligned}\right\}\tag{6.14}$$

$$
\begin{aligned}
&+\frac{n\pi}{4\alpha^2}\left[n\pi+2\alpha(1-\nu-\delta_1)-\frac{n^2\pi^2}{2\alpha}\delta_2\right]B_nR_1^{-\frac{n\pi}{2\alpha}}+\\
&+\frac{n\pi+4\alpha}{4\alpha^2}\left[n\pi+2\alpha(1+\nu+\delta_1)+\frac{n^2\pi^2}{2\alpha}\delta_2\right]C_nR_1^{\frac{n\pi}{2\alpha}+2}+\\
&+\frac{n\pi-4\alpha}{4\alpha^2}\left[n\pi-2\alpha(1+\nu+\delta_1)-\frac{n^2\pi^2}{2\alpha}\delta_2\right]D_nR_1^{-\frac{n\pi}{2\alpha}+2}=\\
&=-\lambda_nR_1\left(\delta_1+\frac{n^2\pi^2}{4\alpha^2}\delta_2\right)\\
&\frac{n\pi}{4\alpha^2}\left[n\pi-2\alpha(1-\nu+\delta_1')-\frac{n^2\pi^2}{2\alpha}\delta_2'\right]A_nR^{\frac{n\pi}{2\alpha}}+\\
&+\frac{n\pi}{4\alpha^2}\left[n\pi+2\alpha(1-\nu+\delta_1')+\frac{n^2\pi^2}{2\alpha}\delta_2'\right]B_nR^{-\frac{n\pi}{2\alpha}}+\\
&+\frac{n\pi+4\alpha}{4\alpha^2}\left[n\pi+2\alpha(1+\nu-\delta_1')-\frac{n^2\pi^2}{2\alpha}\delta_2'\right]C_nR^{\frac{n\pi+4\alpha}{2\alpha}}+\\
&+\frac{n\pi-4\alpha}{4\alpha^2}\left[n\pi-2\alpha(1+\nu-\delta_1')+\frac{n^2\pi^2}{2\alpha}\delta_2'\right]D_nR^{-\frac{n\pi}{2\alpha}+2}=\\
&=\mu_nR\left(\delta_1'+\frac{n^2\pi^2}{4\alpha^2}\delta_2'\right)
\end{aligned}
\qquad (6.14)\ \text{cont'd}
$$

The solution of system (6.14) is

$$
\begin{aligned}
A_n&=\frac{R_1^{-\frac{n\pi}{2\alpha}}}{\Delta}\left\{(k^{\frac{n\pi}{2\alpha}+2}-k^{-\frac{n\pi}{2\alpha}})\left[b_6\lambda_nR_1\left(\delta_1+\frac{n^2\pi^2}{4\alpha^2}\delta_2\right)+\right.\right.\\
&\left.+a_6\mu_nR\left(\delta_1'+\frac{n^2\pi^2}{4\alpha^2}\delta_2'\right)\right]-(k^{-\frac{n\pi}{2\alpha}+2}-k^{-\frac{n\pi}{2\alpha}})\times\\
&\left.\times\left[b_7\lambda_nR_1\left(\delta_1+\frac{n^2\pi^2}{4\alpha^2}\delta_2\right)+a_7\mu_nR\left(\delta_1'+\frac{n^2\pi^2}{4\alpha^2}\delta_2'\right)\right]\right\},\\
B_n&=\frac{R_1^{\frac{n\pi}{2\alpha}}}{\Delta}\left\{(k^{\frac{n\pi}{2\alpha}}-k^{\frac{n\pi}{2\alpha}+2})\left[b_6\lambda_nR_1\left(\delta_1+\frac{n^2\pi^2}{4\alpha^2}\delta_2\right)+\right.\right.\\
&\left.+a_6\mu_nR\left(\delta_1'+\frac{n^2\pi^2}{4\alpha^2}\delta_2'\right)\right]-(k^{\frac{n\pi}{2\alpha}}-k^{-\frac{n\pi}{2\alpha}+2})\times\\
&\left.\times\left[b_7\lambda_nR_1\left(\delta_1+\frac{n^2\pi^2}{4\alpha^2}\delta_2\right)+a_7\mu_nR\left(\delta_1'+\frac{n^2\pi^2}{4\alpha^2}\delta_2'\right)\right]\right\}\\
C_n&=\frac{R_1^{-\frac{n\pi}{2\alpha}-2}}{\Delta}(k^{-\frac{n\pi}{2\alpha}}-k^{\frac{n\pi}{\alpha}})\left[b_6\lambda_nR_1\left(\delta_1+\frac{n^2\pi^2}{4\alpha^2}\delta_2\right)+\right.\\
&\left.+a_6\mu_nR\left(\delta_1'+\frac{n^2\pi^2}{4\alpha^2}\delta_2'\right)\right]\\
D_n&=\frac{R_1^{\frac{n\pi}{2\alpha}-2}}{\Delta}(k^{\frac{n\pi}{2\alpha}}-k^{-\frac{n\pi}{2\alpha}})\left[b_7\lambda_nR_1\left(\delta_1+\frac{n^2\pi^2}{4\alpha^2}\delta_2\right)+\right.\\
&\left.+a_7\mu_nR\left(\delta_1'+\frac{n^2\pi^2}{4\alpha^2}\delta_2'\right)\right]
\end{aligned}
\qquad (6.15)
$$

where

$$\Delta = a_6 b_7 - a_7 b_6, \qquad k = \frac{R}{R_1}$$

$$a_7 = a_1 k^{\frac{n\pi}{2\alpha}+2} + a_2 k^{\frac{n\pi}{2\alpha}} + a_3 k^{-\frac{n\pi}{2\alpha}}$$

$$a_6 = a_1 k^{-\frac{n\pi}{2\alpha}+2} + a_4 k^{\frac{n\pi}{2\alpha}} + a_5 k^{-\frac{n\pi}{2\alpha}}$$

$$b_7 = b_1 k^{\frac{n\pi}{\alpha}+2} + b_2 k^2 + b_3$$

$$b_6 = b_4 k^{-\frac{n\pi}{\alpha}+2} + b_5 k^2 + b_3$$

$$a_1 = -\frac{n\pi}{\alpha}(1 - \nu - \delta_1) + \frac{n^3\pi^3}{4\alpha^3}\delta_2$$

$$a_2 = -2\left(\frac{n\pi}{2\alpha} + 1\right)(1 + \nu + \delta_1) - \frac{n^2\pi^2}{2\alpha^2}\left(\frac{n\pi}{2\alpha} + 1\right)\delta_2$$

$$a_3 = 2(1 + \nu + \delta_1) + \frac{2n\pi}{\alpha} + \frac{n^2\pi^2}{2\alpha^2}\delta_2,$$

$$a_4 = \frac{2n\pi}{\alpha} - \frac{n^2\pi^2}{2\alpha^2}\delta_2 - 2(1 + \nu + \delta_1),$$

$$a_5 = -2\left(\frac{n\pi}{2\alpha} - 1\right)(1 + \nu + \delta_1) + \frac{n^2\pi^2}{2\alpha^2}\left(1 - \frac{n\pi}{2\alpha}\right)\delta_2,$$

$$b_1 = -2(1 + \nu - \delta_1') - \frac{2n\pi}{\alpha} + \frac{n^2\pi^2}{2\alpha^2}\delta_2',$$

$$b_2 = 2\left(1 + \frac{n\pi}{2\alpha}\right)(1 + \nu - \delta_1') - \frac{n^2\pi^2}{2\alpha^2}\left(1 + \frac{n\pi}{2\alpha}\right)\delta_2',$$

$$b_3 = \frac{n\pi}{\alpha}(1 - \nu + \delta_1') + \frac{n^3\pi^3}{4\alpha^3}\delta_2',$$

$$b_4 = -\frac{2n\pi}{\alpha} - \frac{n^2\pi^2}{2\alpha^2}\delta_2' + 2(1 + \nu - \delta_1'),$$

$$b_5 = 2\left(\frac{n\pi}{2\alpha} - 1\right)(1 + \nu - \delta_1') + \frac{n^2\pi^2}{2\alpha^2}\left(1 - \frac{n\pi}{2\alpha}\right)\delta_2'. \qquad (6.16)$$

The problem has thus been solved. Substituting in (6.15) and (6.16) $\delta_1 \to \infty$ and $\delta_2 \to \infty$ or $\delta_1' \to \infty$ and $\delta_2' \to \infty$, we obtain the coefficients A_n, B_n, C_n, D_n for the case when the curved plate edges are rigidly clamped.

We shall show in conclusion that the bending moments L_n vanish at the end sections $\theta = \pm\alpha$ of the stiffening ribs.

In fact, according to (2.10) and (2.14),

$$L_n = A\delta\omega_n = A\left(\frac{\partial\theta_n}{\partial s} + \theta_n \frac{1}{\varrho}\right), \qquad (6.17)$$

where $\varrho = R$ for the inner rib, and $\varrho = R_1$ for the outer rib.

Since

$$\theta_n = -\frac{\partial w}{\partial s}\Big|_{r=\varrho}, \qquad \theta_\tau = \frac{\partial w}{\partial n}\Big|_{r=\varrho},$$

we obtain from (6.17) for $\theta = \pm\alpha$

$$L_n = -A\left(\frac{1}{\varrho^2}\cdot\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{\varrho}\cdot\frac{\partial w}{\partial r}\right)_{\substack{r=\varrho \\ \theta=\pm\alpha}}. \tag{6.18}$$

But according to (6.2),

$$\frac{\partial w}{\partial r} = \frac{\partial^2 w}{\partial\theta^2} = 0 \text{ at } \theta = \pm\alpha \tag{6.19}$$

for all values of r, including $r = \varrho$.

Substituting (6.19) in (6.18), we obtain

$$L_n = 0. \tag{6.20}$$

This proves our assertion.

Only the transverse reactions V_b and twisting moments L_τ act at the end sections of the stiffening ribs. We obtain from (2.10), (2.14), and (2.19):

$$L_\tau = C\delta\omega_\tau = C\left(\frac{\partial\theta_\tau}{\partial s} - \frac{1}{\varrho}\theta_n\right) = C\left(\frac{1}{\varrho}\cdot\frac{\partial^2 w}{\partial r\partial\theta} + \frac{1}{\varrho^2}\frac{\partial w}{\partial\theta}\right)$$

at $\theta = \pm\alpha$,

$$V_b = \frac{d}{ds}L_n + \frac{1}{\varrho}L_\tau = -A\left(\frac{1}{\varrho^3}\cdot\frac{\partial^3 w}{\partial\theta^3} + \frac{1}{\varrho^2}\cdot\frac{\partial^2 w}{\partial r\partial\theta}\right) + \tag{6.21}$$

$$+ C\left(\frac{1}{\varrho^2}\cdot\frac{\partial^2 w}{\partial r\partial\vartheta} + \frac{1}{\varrho^3}\cdot\frac{\partial w}{\partial\vartheta}\right).$$

From the boundary condition $w = 0$ ((6.3)) it follows also that

$$\frac{\partial w}{\partial\vartheta} = \frac{\partial^2 w}{\partial\vartheta^2} = \frac{\partial^3 w}{\partial\vartheta^3} = 0 \text{ at } r = \varrho. \tag{6.22}$$

Hence, (6.21) becomes

$$L_\tau = \frac{1}{\varrho}C\frac{\partial^2 w}{\partial r\partial\vartheta}, \qquad V_b = (C - A)\frac{1}{\varrho^2}\cdot\frac{\partial^2 w}{\partial r\partial\theta} \text{ at } \theta = \pm\alpha. \tag{6.23}$$

Bending of sectorial plates with reinforced curved edges

Consider a sectorial plate whose central angle is $2\alpha < \pi$. The plate is hinged at all edges (Figure 84). In this case $R = 0$, i. e., $k = \frac{R}{R_1} = 0$, and

(6.15) becomes

$$\left.\begin{aligned} A_n = -R_1^2 C_n = \frac{\lambda_n(\delta_1 + \beta_n^2\delta_2)}{2[(1+\nu+\delta_1)+2\beta_n+\beta_n^2\delta_2]}\, R_1^{1-\beta_n} \\ B_n = D_n = 0 \end{aligned}\right\}, \tag{6.24}$$

where

$$\beta_n = \frac{n\pi}{2\alpha},$$

and the coefficients λ_n are given by (6.11).

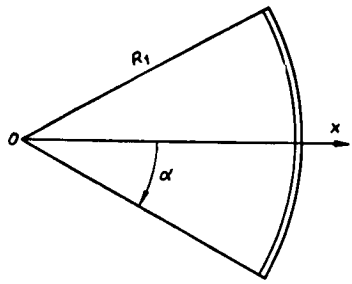


FIGURE 84.

Substituting (6.24) in (6.1), we find the total deflection of a sectorial plate with reinforced curved edge:

$$w = w^0(r,\ \theta) + \left(1 - \frac{r^2}{R_1^2}\right)\sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{2\alpha}} \sin\frac{n\pi(\alpha+\theta)}{2\alpha}. \tag{6.25}$$

If the plate is rigidly clamped along the curved edge, i.e., $\delta_1 = \infty$ and $\delta_2 = \infty$, then

$$A_n = \frac{1}{2}\lambda_n R_1^{1-\beta_n}.$$

Using the numerous solutions $w^0(r, \theta)$ for hinged sectorial plates, given by Galerkin /20/, we easily obtain the total deflections $w(r, \theta)$ of a plate with reinforced curved edge.

The additional bending moments in the plate are

$$\begin{aligned} M_r = -D\sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{2\alpha}-2}\Big\{\beta_n(1-\nu)(\beta_n-1) - [(1-\nu)\beta_n^2 + \\ + (3+\nu)\beta_n + 2(1+\nu)]\frac{r^2}{R_1^2}\Big\}\sin\frac{n\pi(\alpha+\theta)}{2\alpha} \end{aligned} \tag{6.26}$$

$$M_\theta = -D\sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{2\alpha}-2}\left\{\beta_n(1-\nu)(1-\beta_n)+[(1-\nu)\beta_n^2- -(1+3\nu)\beta_n-2(1+\nu)]\frac{r^2}{R_1^2}\right\}\sin\frac{n\pi(\alpha+\theta)}{2\alpha} \quad (6.26)\ (\text{cont'd})$$

§ 2. BENDING OF A SECTORIAL PLATE WITH REINFORCED CURVED EDGE FREE OF EXTERNAL LOAD

A sectorial plate whose central angle is $2\alpha < \pi$ is simply supported along the radial edges $\theta = \pm\alpha$. The curved edge $r = R_1$ is reinforced by a thin elastic rib of uniform cross section (cf. Figure 84). The load is arbitrary.

The equation of the middle surface of the plate can be represented in the form

$$w = w^0 + w^1 = w^0(r, \theta) + \sum_{n=1}^{\infty}\left(A_n r^{\frac{n\pi}{2\alpha}} + C_n r^{\frac{n\pi}{2\alpha}+2}\right)\sin\frac{n\pi(\alpha+\theta)}{2\alpha}, \quad (6.27)$$

where $w^0(r, \theta)$ are the basic deflections, considered as known /20/, of the nonreinforced plate under the same load. The boundary conditions (6.2) (simple support along the edges $\theta = \pm\alpha$) are satisfied automatically.

The coefficients A_n and C_n are determined from the boundary conditions (2.359) for the reinforced curved edge:

at $r = R_1$

$$\left.\begin{aligned}&(\delta_1+\nu)\frac{\partial w}{\partial r}+R_1\frac{\partial^2 w}{\partial r^2}+\frac{\delta_1+\delta_2+\nu}{R_1}\cdot\frac{\partial^2 w}{\partial\theta^2}-\delta_2\frac{\partial^3 w}{\partial r\partial\theta^2}=0\\&\frac{\partial w}{\partial r}-R_1\frac{\partial^2 w}{\partial r^2}-(\delta_2+\nu-3)\frac{1}{R_1}\cdot\frac{\partial^2 w}{\partial\theta^2}-R_1^2\frac{\partial^3 w}{\partial r^3}+\\&+(\delta_1+\delta_2+\nu-2)\frac{\partial^3 w}{\partial r\partial\theta^2}+\delta_1\frac{1}{R_1}\cdot\frac{\partial^4 w}{\partial\theta^4}=0\end{aligned}\right\}. \quad (6.28)$$

By hypothesis the function $w^0(r, \theta)$ satisfies the boundary conditions that no bending moments or generalized shearing forces act at the edge $r = R_1$ [cf. (1.16)]:

$$\left.\begin{aligned}&\frac{\partial^2 w^0}{\partial r^2}+\nu\left(\frac{1}{r}\cdot\frac{\partial w^0}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 w^0}{\partial\theta^2}\right)=0\\&\frac{\partial}{\partial r}\left(\frac{\partial^2 w^0}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial w^0}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 w^0}{\partial\theta^2}\right)+\\&+(1-\nu)\frac{1}{r}\cdot\frac{\partial}{\partial\theta}\left(\frac{1}{r}\cdot\frac{\partial^2 w^0}{\partial r\partial\theta}-\frac{1}{r^2}\cdot\frac{\partial w^0}{\partial\theta}\right)=0\end{aligned}\right\}. \quad (6.29)$$

Substituting (6.27) in (6.28), and taking (6.29) into account, we obtain the following boundary conditions for the additional deflections w^1 at $r = R_1$:

$$\left.\begin{aligned}
&(\delta_1+\nu)\frac{\partial w^1}{\partial r}+R_1\frac{\partial^2 w^1}{\partial r^2}+(\delta_1+\delta_2+\nu)\frac{1}{R_1}\frac{\partial^2 w^1}{\partial\theta^2}-\delta_2\frac{\partial^3 w^1}{\partial r\partial\theta^2}=\\
&=f_1(\theta)=-\delta_1\left(\frac{\partial w^0}{\partial r}+\frac{1}{R_1}\cdot\frac{\partial^2 w^0}{\partial\theta^2}\right)-\delta_2\left(\frac{1}{R_1}\cdot\frac{\partial^2 w^0}{\partial\theta^2}-\frac{\partial^3 w^0}{\partial r\partial\theta^2}\right)\\
&\frac{\partial w^1}{\partial r}-R_1\frac{\partial^2 w^1}{\partial r^2}-(\delta_2+\nu-3)\frac{1}{R_1}\cdot\frac{\partial^2 w^1}{\partial\theta^2}-R_1^2\frac{\partial^3 w^1}{\partial r^3}+\\
&+(\delta_1+\delta_2+\nu-2)\frac{\partial^3 w^1}{\partial r\partial\theta^2}+\frac{\delta_1}{R_1}\cdot\frac{\partial^4 w^1}{\partial\theta^4}=-f_2(\theta)=\\
&=-\delta_1\left(\frac{\partial^3 w^0}{\partial r\partial\theta^2}+\frac{1}{R_1}\cdot\frac{\partial_4 w^\circ}{\partial\theta^4}\right)+\delta_2\left(\frac{1}{R_1}\cdot\frac{\partial^2 w^0}{\partial\theta^2}-\frac{\partial^3 w^0}{\partial r\partial\theta^2}\right)
\end{aligned}\right\}. \tag{6.30}$$

Expanding the known functions entering in the right-hand sides of (6.30) in sine series:

$$\left.\begin{aligned}
\frac{1}{R_1}\left(\frac{\partial^2 w^0}{\partial\theta^2}\right)_{r=R_1}&=\sum_{n=1}^{\infty}\mu_n\sin\frac{n\pi(\alpha+\theta)}{2\alpha}\\
\left(\frac{\partial w^0}{\partial r}\right)_{r=R_1}&=\sum_{n=1}^{\infty}\lambda_n\sin\frac{n\pi(\alpha+\theta)}{2\alpha}
\end{aligned}\right\}, \tag{6.31}$$

we obtain

$$\left.\begin{aligned}
f_1(\theta)&=-\sum_{n=1}^{\infty}\left[\delta_1(\lambda_n+\mu_n)+\delta_2\left(\mu_n+\frac{n^2\pi^2}{4\alpha^2}\lambda_n\right)\right]\sin\frac{n\pi(\alpha+\theta)}{2\alpha}\\
f_2(\theta)&=-\sum_{n=1}^{\infty}\left[\delta_1(\lambda_n+\mu_n)\frac{n^2\pi^2}{4\alpha^2}+\delta_2\left(\mu_n+\frac{n^2\pi^2}{4\alpha^2}\lambda_n\right)\right]\sin\frac{n\pi(\alpha+\theta)}{2\alpha}
\end{aligned}\right\}. \tag{6.32}$$

Inserting (6.27) and (6.32) into (6.30), we obtain the following two equations for the coefficients A_n and C_n:

$$\left.\begin{aligned}
&C_nR_1^{\beta_n+2}[(1-\nu)\beta_n+2(1+\nu)-\delta_1(\beta_n-2)+\delta_2\beta_n^2](\beta_n+1)+\\
&+A_nR_1^{\beta_n}[(1-\nu)-\delta_1+\delta_2\beta_n](\beta_n-1)\beta_n=-R_1[\delta_1(\lambda_n+\\
&+\mu_n)+\delta_2(\mu_n+\beta_n^2\lambda_n)],\\
&C_nR_1^{\beta_n+2}[(\beta_n-\nu\beta_n-4)+\delta_1\beta_n(\beta_n-2)-\delta_2\beta_n](\beta_n+1)\beta_n+\\
&+A_nR_1^{\beta_n}[(1-\nu)+\beta_n\delta_1-\delta_2]\beta_n^2(\beta_n-1)=\\
&=R_1[\delta_1(\lambda_n+\mu_n)\beta_n^2+\delta_2(\mu_n+\lambda_n\beta_n^2)]
\end{aligned}\right\}, \tag{6.33}$$

where $\beta_n = \frac{n\pi}{2a}$.

Solving system (6.33), we obtain

$$\left.\begin{aligned} A_n &= \frac{\beta_n + 1}{R_1^{\beta_n - 1}\Delta_*}\{[(1-\nu)\beta_n + 2(1+\nu) - \delta_1(\beta_n - 2) + \\ &+ \delta_2\beta_n^2]Z_2 - \beta_n[(\beta_n - \nu\beta_n - 4) + \delta_1\beta_n(\beta_n - 2) - \delta_2\beta_n]Z_1\} \\ C_n &= \frac{\beta_n(\beta_n - 1)}{R_1^{\beta_n + 1}\Delta_*}\{\beta_n(1 - \nu + \delta_1\beta_n - \delta_2)Z_1 - \\ &- (1 - \nu - \delta_1 + \delta_2\beta_n)Z_2\} \end{aligned}\right\}, \qquad (6.34)$$

where

$$Z_1 = -\delta_1(\lambda_n + \mu_n) - \delta_2(\mu_n + \beta_n^2\lambda_n),$$

$$Z_2 = \delta_1(\lambda_n + \mu_n)\beta_n^2 + \delta_2(\mu_n + \lambda_n\beta_n^2);$$

$$\Delta_* = 2\beta_n^2(\beta_n^2 - 1)\{(1-\nu)(3+\nu) + (\delta_1 + \delta_2)(2\beta_n - 1 - \nu) +$$

$$+ \delta_1\delta_2(\beta_n^2 - 1)\}.$$

The problem has thus been solved.

As in the preceding section, it can be shown that no bending moments, but only transverse support reactions and twisting moments act at the end sections of the stiffening rib.

§ 3. BENDING OF AN ARBITRARILY LOADED CIRCULAR PLATE WITH SUPPORTING RIB

An arbitrarily loaded circular plate of radius R_1 is reinforced along the contour by a thin supporting rib. We determine the additional deflections, due to the supporting ring, by representing the elastic surface in Clebsch's form (/136/, Chapter VII)

$$w = w^0 + w^1 = w^0 + \sum_{n=0}^{\infty}(A_n r^n + C_n r^{n+2})\cos n\theta + \sum_{n=1}^{\infty}(A_n' r^n + C_n' r^{n+2})\sin n\theta, \qquad (6.35)$$

where w^0 are the deflections of the simply-supported ribless plate under the same load; (r, θ) are polar coordinates with origin at the plate center.

We determine the coefficients A_n, C_n, A_n', C_n' from the boundary conditions (2.364):

$$w = 0, \qquad (\delta_1 + \nu)\frac{\partial w}{\partial r} + R_1\frac{\partial^2 w}{\partial r^2} - \delta_2\frac{\partial^3 w}{\partial r\partial\theta^2} = 0. \qquad (6.36)$$

By hypothesis the function w^0 satisfies the boundary conditions (6.6) for simple support along $r = R_1$. We thus again obtain condition (6.7) for the additional deflections w'.

Writing

$$\left(\frac{\partial w^0}{\partial r}\right)_{r=R_1} = \sum_{n=1}^{\infty} \lambda_n' \sin n\theta + \sum_{n=0}^{\infty} \lambda_n \cos n\theta, \tag{6.37}$$

we obtain

$$\left(\frac{\partial^2 w^0}{\partial r \partial \theta^2}\right)_{r=R_1} = -\sum_{n=1}^{\infty} n^2 \lambda_n' \sin n\theta - \sum_{n=0}^{\infty} n^2 \lambda_n \cos n\theta, \tag{6.38}$$

where the coefficients λ_n and λ_n' are known. Substituting (6.37), (6.38), and (6.35) in (6.7), we obtain the following equations for determining the coefficients A_n, C_n, A_n', C_n':

$$\begin{gathered} A_n + C_n R_1^2 = 0, \\ A_n R_1^n n(n + \nu - 1 + \delta_1 + n^2\delta_2) + C_n R_1^{n+2}(n+2)(n+1+\nu + \\ + \delta_1 + n^2\delta_2) = -R_1(\delta_1 + n^2\delta_2)\lambda_n. \end{gathered} \tag{6.39}$$

Replacing λ_n by λ_n', we obtain a similar system of equations for the coefficients A_n' and C_n'.

Solving system (6.39), we obtain

$$A_n R_1^n = -C_n R_1^{n+2} = \frac{R_1(\delta_1 + n^2\delta_2)\lambda_n}{2(2n + 1 + \nu + \delta_1 + n^2\delta_2)}, \tag{6.40}$$

and similarly

$$A_n' R_1^n = -C_n' R_1^{n+2} = \frac{R_1(\delta_1 + n^2\delta_2)\lambda_n'}{2(2n + 1 + \nu + \delta_1 + n^2\delta_2)}. \tag{6.41}$$

Substituting (6.40) and (6.41) in (6.35), we obtain the additional deflections

$$\left.\begin{gathered} w^1 = R_1 \sum_{n=0}^{\infty} (\lambda_n \cos n\theta + \lambda_n' \sin n\theta) v_n \\ \text{where} \quad v_n = \frac{(\delta_1 + n^2\delta_2)}{2(2n + 1 + \nu + \delta_1 + n^2\delta_2)} \left[\left(\frac{r}{R_1}\right)^n - \left(\frac{r}{R_1}\right)^{n+2}\right] \end{gathered}\right\}. \tag{6.42}$$

Adding (6.42) to any solution w^0 for a simply supported plate under the same load, we obtain the equation of the middle surface of a plate reinforced by a supporting rib.

Circular plate subjected to a concentrated load

Consider as example the problem of bending of a circular plate with supporting rib of radius R_1, subjected to a concentrated force applied at an

arbitrary point $R,0$ (Figure 85). The corresponding deflections of a simply supported plate without supporting rib are /203/

$$w^0 = F_0 + \sum_{n=1}^{\infty} F_n \cos n\theta, \tag{6.43}$$

where

for $R < r < R_1$

$$\left.\begin{aligned}
F_0 &= \frac{P}{8\pi D}\left[\frac{(3+\nu)R_1^2-(1-\nu)R^2}{2(1+\nu)R_1^2}\times\right.\\
&\left.\times(R_1^2-r^2)+(R^2+r^2)\ln\frac{r}{R_1}\right]\\
F_1 &= \frac{PR}{8\pi D}\left[\frac{1+\nu}{3+\nu}\cdot\frac{R^2-R_1^2}{R_1^2}r-2r\ln\frac{r}{R_1}+\right.\\
&\left.+\frac{2R_1^2(1+\nu)+(1-\nu)R^2}{2(3+\nu)R_1^4}r^3-\frac{R^2}{2r}\right]\\
F_n &= \frac{PR^n}{8\pi Dn(n-1)}\left\{\frac{r^n}{(2n+1+\nu)R_1^{2n}}\left[(1+\nu)(n-1)R^2-\right.\right.\\
&-(3+\nu)nR_1^2+(1+\nu)(n-1)r^2+\frac{(n-1)n}{n+1}\times\\
&\left.\left.\times\frac{R^2r^2}{R_1^2}(1-\nu)\right]+\frac{1}{r^n}\left(r^2-\frac{n-1}{n+1}R^2\right)\right\}
\end{aligned}\right\}; \tag{6.44}$$

for $0 < r < R$

$$\left.\begin{aligned}
F_0 &= \frac{P}{8\pi D}\left[\frac{(3+\nu)R_1^2-(1-\nu)r^2}{2(1+\nu)R_1^2}(R_1^2-R^2)+\right.\\
&\left.+(R^2+r^2)\ln\frac{R}{R_1}\right]\\
F_1 &= \frac{PR}{8\pi D}\left[\frac{1+\nu}{3+\nu}\cdot\frac{R^2-R_1^2}{R_1^2}r-2r\ln\frac{R}{R_1}+\right.\\
&\left.+\frac{2R_1^2R^2(1+\nu)+R^4(1-\nu)-R_1^4(3+\nu)}{2(3+\nu)R_1^4R^2}r^3\right]\\
F_n &= \frac{PR^n}{8\pi Dn(n-1)}\left\{\frac{r^n}{R_1^{2n}}\left[\frac{(1+\nu)(n-1)R^2-(3+\nu)nR_1^2}{1+2n+\nu}+\right.\right.\\
&\left.+\frac{R_1^{2n}}{R^{2n-2}}\right]+(n-1)\frac{r^{n+2}}{R_1^{2n}}\left[\frac{(1+\nu)(n+1)R_1^2+n(1-\nu)R^2}{(2n+1+\nu)(n+1)R_1^2}-\right.\\
&\left.\left.-\frac{1}{n+1}\cdot\frac{R_1^{2n}}{R^{2n}}\right]\right\}
\end{aligned}\right\} \tag{6.45}$$

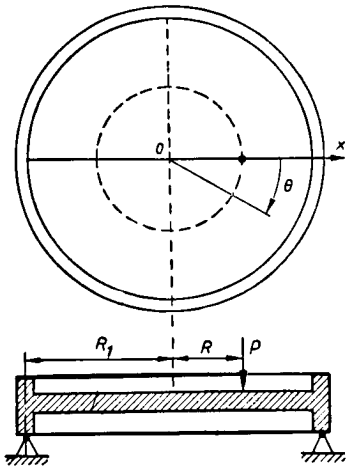


FIGURE 85.

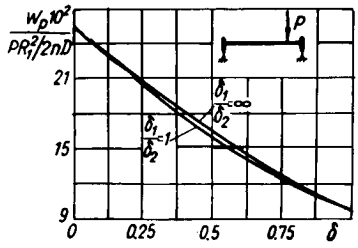


FIGURE 86.

Differentiating (6.43) by r, we obtain $\left(\frac{\partial w^0}{\partial r}\right)_{r=R_1}$ and find that according to (6.37)

$$\lambda_n' = 0, \quad \lambda_n = \left(\frac{dF_n}{dr}\right)_{r=R_1} \quad (n > 0), \tag{6.46}$$

i. e.,

$$\lambda_0 = \frac{P}{4\pi D} \cdot \frac{R^2 - R_1^2}{R_1(1+\nu)}, \qquad \lambda_n = \frac{PR^n(R^2 - R_1^2)}{2\pi D(2n+1+\nu)R_1^{n+1}}. \quad (n > 1). \tag{6.47}$$

Substituting (6.46) and (6.47) in (6.42), we obtain the additional deflections w^1, and then the total deflections w ((6.35)):

$$w = N_0 + \sum_{n=1}^{\infty} N_n \cos n\vartheta.$$

Here

$$N_n = F_n + \frac{(\delta_1 + n^2\delta_2)\lambda_n R_1}{2(2n+1+\nu+\delta_1+n^2\delta_2)}\left[\left(\frac{r}{R_1}\right)^n - \left(\frac{r}{R_1}\right)^{n+2}\right]\cos n\vartheta. \tag{6.48}$$
$$(n > 0).$$

In the particular case $\delta_1 = \delta_2 = \infty$, we obtain from (6.48) the solution for a plate rigidly clamped at the edge (/136/, p. 256).

As example*, the deflections w_p due to a force P were determined as functions of δ_1 and δ_2 for $\frac{R_1}{R} = 2$. The results of the calculations showed that in the case considered the flexural rigidity (δ_1) is most important, while the torsional rigidity (δ_2) has a negligible effect on the deflections w_p. The

* The calculations were performed by S. I. Sidorenko.

results are given in Figure 86 which shows w_p as a function of $\delta = \frac{\delta_1}{\delta_1 + 2}$ for $\frac{\delta_1}{\delta_2} = 1$ and $\frac{\delta_1}{\delta_2} = \infty$.

§ 4. AXISYMMETRICAL BENDING OF AN ANNULAR PLATE OF VARIABLE THICKNESS WITH REINFORCED EDGES

Consider the axisymmetrical bending of a thin isotropic annular plate of variable thickness, whose edges are reinforced by thin elastic ribs of uniform cross section, made from a different material. The radii of the axial lines of the stiffening ribs are R_1 and R. It is assumed that one of the principal axes of inertia of each rib cross section lies in the middle plane of the plate.

The basic equation of bending of a plate of variable thickness, subjected to a transverse load, is /55/

$$D\frac{d}{dr}\left(\frac{d\vartheta}{dr} + \frac{\vartheta}{r}\right) + \frac{dD}{dr}\left(\frac{d\vartheta}{dr} + \nu\frac{\vartheta}{r}\right) = -Q_r \tag{6.49}$$

where $\vartheta = -\frac{dw}{dr}$; w are the deflections of points of the middle plane; $D = D(r)$ is the variable flexural rigidity of the plate; $Q_r = -\frac{1}{r}\int qrdr + C$ is the shearing force per unit length of the circular section of radius r; ν is Poisson's ratio; $q = q(r)$ is the transverse load per unit area. Details on studies of the bending of thin plates of variable thickness are given in (/56/, /57/).

The general solution of (6.49) can be represented in the form

$$\vartheta = \vartheta_0 + \vartheta_1, \tag{6.50}$$

where

$$\vartheta_0 = -\frac{dw^0}{dr}; \qquad \vartheta_1 = -\frac{dw^1}{dr}; \tag{6.50a}$$

w^0 are the basic deflections considered as known of the nonreinforced plate under the same load q; w^1 are the additional deflections due to the stiffening ribs. Expressions for the functions w^0 or ϑ_0 for different loads acting on circular and annular plates of variable thickness can be found in /55/.

The function ϑ_1 must satisfy the homogeneous equation corresponding to (6.49), and can be written in the form

$$\vartheta_1 = C_1\vartheta^{(1)} + C_2\vartheta^{(2)}, \tag{6.51}$$

where $\vartheta^{(1)}$ and $\vartheta^{(2)}$ are independent particular solutions of (6.49) for $Q_r = 0$. In (/55/, §§ 10 to 15) these are given as hypergeometric functions when the rigidity of a plate of nonuniform thickness varies in a very general manner; $\vartheta^{(1)}$ and $\vartheta^{(2)}$ are integration constants which have to be determined from the boundary conditions (2.366) and (2.367).

Hence, at the outer edge $r = R_1$,

$$(\delta_1 + \nu)\,\vartheta + R_1 \frac{d\vartheta}{dr} = m_0 \frac{R_1}{D}, \tag{6.52}$$

and at the inner edge $r = R$,

$$(\delta_1' - \nu)\,\vartheta - R \frac{d\vartheta}{dr} = -m_0' \frac{R}{D}, \tag{6.53}$$

where

$$\delta_1 = \frac{A_1}{R_1 (D)_{r=R_1}}, \qquad \delta_1' = \frac{A_1'}{R (D)_{r=R}}, \tag{6.54}$$

A_1, A_1' are the flexural rigidities of the outer and inner ribs; m_0 and m_0' are the given applied bending moments.

By hypothesis, the function ϑ_0 satisfies the following relationships:

$$\nu\vartheta_0 + R_1 \frac{d\vartheta_0}{dr} = m_0 \frac{R_1}{D} \quad \text{at} \quad r = R_1, \tag{6.55}$$

$$\nu\vartheta_0 + R \frac{d\vartheta_0}{dr} = m_0' \frac{R}{D} \quad \text{at} \quad r = R. \tag{6.56}$$

Substituting (6.50) in (6.52) and (6.53), and taking (6.55), (6.56), into account, we obtain the boundary conditions for the derivative ϑ_1 of the additional deflection w^1:

$$\left.\begin{aligned} (\delta_1 + \nu)\,\vartheta_1 + R_1 \frac{d\vartheta_1}{dr} &= -\delta_1 \vartheta_0 \quad \text{at} \quad r = R_1, \\ (\delta_1' - \nu)\,\vartheta_1 - R \frac{d\vartheta_1}{dr} &= -\delta_1' \vartheta_0 \quad \text{at} \quad r = R. \end{aligned}\right\} \tag{6.57}$$

Substitution of (6.51) in (6.57) yields a system of algebraic equations in the unknown coefficients C_1 and C_2, whose solution is

$$\left.\begin{aligned} C_1 &= \frac{1}{\Delta}\left\{\delta_1' (\vartheta_0)_{r=R}\left[\left(\delta_1 + \nu + R_1 \frac{d}{dr}\right)\vartheta^{(2)}\right]_{r=R_1} - \right. \\ &\quad \left. - \delta_1 (\vartheta_0)_{r=R_1}\left[\left(\delta_1' - \nu - R \frac{d}{dr}\right)\vartheta^{(2)}\right]_{r=R}\right\} \\ C_2 &= \frac{1}{\Delta}\left\{-\delta_1' (\vartheta_0)_{r=R}\left[\left(\delta_1 + \nu + R_1 \frac{d}{dr}\right)\vartheta^{(1)}\right]_{r=R_1} + \right. \\ &\quad \left. + \delta_1 (\vartheta_0)_{r=R_1}\left[\left(\delta_1' - \nu - R \frac{d}{dr}\right)\vartheta^{(1)}\right]_{r=R}\right\} \end{aligned}\right\}, \tag{6.58}$$

where

$$\begin{aligned} \Delta = &\left[\left(\delta_1 + \nu + R_1 \frac{d}{dr}\right)\vartheta^{(1)}\right]_{r=R_1} \cdot \left[\left(\delta_1' - \nu - R \frac{d}{dr}\right)\vartheta^{(2)}\right]_{r=R} - \\ &- \left[\left(\delta_1 + \nu + R_1 \frac{d}{dr}\right)\vartheta^{(2)}\right]_{r=R_1} \cdot \left[\left(\delta_1' - \nu - R \frac{d}{dr}\right)\vartheta^{(1)}\right]_{r=R}. \end{aligned} \tag{6.59}$$

Inserting known expressions for the functions $\vartheta^{(1)}$, $\vartheta^{(2)}$, ϑ_0, we can thus determine the influence of the stiffening ribs on the stresses and deflections of a plate of variable thickness.

The additional deflections are

$$w^1 = -\int \vartheta_1 dr + C_3, \tag{6.60}$$

where C_3 is a constant determined from the condition that $w^1 = 0$ on the support contour.

We shall consider some elementary examples.

Annular plate with reinforced edges supported along the outer contour

Let the rigidity of the plate be

$$D = D_0 r^m, \tag{6.61}$$

where D_0 and m are constants.

The particular solutions $\vartheta^{(1)}$ and $\vartheta^{(2)}$ of the homogeneous equation corresponding to (6.49) are in this case

$$\vartheta^{(1)} = r^{\gamma}, \quad \vartheta^{(2)} = r^{-\beta}, \tag{6.62}$$

where

$$\gamma = \alpha - \frac{m}{2}, \quad \beta = \alpha + \frac{m}{2}, \quad \alpha = \sqrt{1 - m\nu + \left(\frac{m}{2}\right)^2}. \tag{6.63}$$

Substituting (6.62) in (6.58) and (6.59), we obtain the coefficients C_1 and C_2 for an arbitrary axisymmetrical load acting on a plate whose rigidity is given by (6.61):

$$\left.\begin{aligned} C_1 &= \frac{1}{\Delta_1}\left[(\vartheta_0)_{r=R} a_1 \delta_1' - (\vartheta_0)_{r=R_1} a_2 \delta_1\right] \\ C_2 &= \frac{1}{\Delta_1}\left[(\vartheta_0)_{r=R_1} b_1 \delta_1 - (\vartheta_0)_{r=R} b_2 \delta_1'\right] \end{aligned}\right\}, \tag{6.64}$$

where

$$\left.\begin{aligned} \Delta_1 &= (\delta_1 + \nu + \gamma)(\delta_1' - \nu + \beta) R_1^{\gamma} R^{-\beta} - \\ &\quad - (\delta_1 + \nu - \beta)(\delta_1' - \nu - \gamma) R_1^{-\beta} R^{\gamma} \\ a_1 &= (\delta_1 + \nu - \beta) R_1^{-\beta}, \quad a_2 = (\delta_1' - \nu + \beta) R^{-\beta} \\ b_1 &= (\delta_1' - \nu - \gamma) R^{\gamma}, \quad b_2 = (\delta_1 + \nu + \gamma) R_1^{\gamma} \end{aligned}\right\}. \tag{6.65}$$

Consider as example the case when no outer rib is provided ($\delta_1 = 0$). The plate is subjected to bending moments $M_1 = \text{const.}$ along the circle $r = R_1$, and to a total transverse load P along the circle $r = R$ (Figure 87).

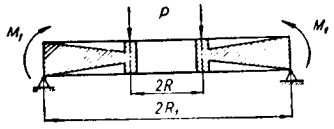


FIGURE 87.

The derivative ϑ_0 of the basic deflection w^0 of the nonreinforced plate is in this case

$$\vartheta_0 = C_1' r^{\gamma} + C_2' r^{-\beta} + \frac{P}{2\pi D_0 r^{m-1} m (1-\nu)}, \tag{6.66}$$

where

$$\left.\begin{aligned} C_1' &= -\frac{R_1^{1-\beta}}{2\pi D_0 m (1-\nu)\Delta_*}\{P(m-1-\nu)(\beta-\nu)\eta^{\beta-1} + \\ &+ [P(m-1-\nu) + 2\pi m(1-\nu)M_1](\nu-\beta)\eta^{2\alpha}\} \\ C_2' &= -\frac{R_1^{1+\gamma}}{2\pi D_0 m(1-\nu)\Delta_*}\{P(m-1-\nu)(\gamma+\nu)\eta^{\beta-1} - \\ &- [P(m-1-\nu) + 2\pi m(1-\nu)M_1](\gamma+\nu)\} \end{aligned}\right\}, \tag{6.67}$$

$$\Delta_* = (\gamma+\nu)(\beta-\nu)(\eta^{2\alpha}-1), \quad \eta = \frac{R_1}{R}. \tag{6.68}$$

Substituting (6.66) in (6.64), we obtain the coefficients C_1 and C_2 of the function ϑ_1 ((6.51)), and thence, according to (6.50), /127/,

$$\vartheta = C_0 r^{\gamma} + C_0^* r^{-\beta} + \frac{P}{2\pi D_0 m (1-\nu) r^{m-1}}. \tag{6.69}$$

where

$$\left.\begin{aligned} C_0 &= -\frac{R_1^{1-\beta}}{2\pi D_0 m (1-\nu)\Delta_0}\{P(m-1-\nu+\delta)(\beta-\nu)\eta^{\beta-1} + \\ &+ [P(m-1-\nu) + 2\pi m(1-\nu)M_1](\nu-\delta-\beta)\eta^{2\alpha}\} \\ C_0^* &= -\frac{R_1^{1+\nu}}{2\pi D_0 m(1-\nu)\Delta_0}\{P(m-1-\nu+\delta)(\gamma+\nu)\eta^{\beta-1} - \\ &- [P(m-1-\nu) + 2\pi m(1-\nu)M_1](\gamma+\nu-\delta)\} \\ \Delta_0 &= (\gamma+\nu)(\beta-\nu)(\eta^{2\alpha}-1) + \delta[(\gamma+\nu)\eta^{2\alpha} + (\beta-\nu)], \\ &\qquad \delta = \delta_1' \end{aligned}\right\}. \tag{6.70}$$

When $P = 0$ and the plate is subjected only to bending moments M_1 applied at its outer edge, (6.69) becomes

$$\vartheta = -\frac{M_1}{D_0\Delta_0}[(\nu - \delta - \beta)\eta^{2\alpha}R_1^{1-\beta}r^{\gamma} - (\nu - \delta + \gamma)R_1^{1+\gamma}r^{-\beta}]. \tag{6.71}$$

The bending moments are

$$M_r = D\left(\frac{d\vartheta}{dr} + \nu\frac{\vartheta}{r}\right), \quad M_\theta = D\left(\frac{\vartheta}{r} + \nu\frac{d\vartheta}{dr}\right). \tag{6.72}$$

The stresses on the upper surface of the plate are:

$$\sigma_r = \frac{6M_r}{h^2}, \quad \sigma_\theta = \frac{6M_\theta}{h^2}, \tag{6.73}$$

where h is the variable plate thickness. Substituting (6.72) and (6.71), we obtain

$$\left.\begin{aligned}
\sigma_r &= \frac{6M_1}{H^2\Delta_0}\left[(\gamma + \nu)(\beta + \delta - \nu)\eta^{2\alpha}\left(\frac{r}{R_1}\right)^{\beta-1-\frac{2m}{3}} - \right. \\
&\left. - (\beta - \nu)(\gamma - \delta + \nu)\left(\frac{R_1}{r}\right)^{1+\gamma+\frac{2m}{3}}\right] \\
\sigma_\theta &= \frac{6M_1}{H^2\Delta_0}\left[(1 + \nu\gamma)(\beta + \delta - \nu)\eta^{2\alpha}\left(\frac{r}{R_1}\right)^{\beta-1-\frac{2m}{3}} - \right. \\
&\left. - (\beta\nu - 1)(\gamma - \delta + \nu)\left(\frac{R_1}{r}\right)^{1+\gamma+\frac{2m}{3}}\right]
\end{aligned}\right\}, \tag{6.74}$$

where H is the thickness of the plate at its outer edge.

It is easily shown by direct substitution that the stress distribution is uniform over the entire plate surface if we assume

$$m = 0, \quad \delta = 1 + \nu. \tag{6.75}$$

In fact, if a plate of uniform thickness is reinforced by a rib whose relative rigidity is $\delta = 1 + \nu$, no stress concentrations will arise in it, and (6.74) becomes

$$\sigma_r = \sigma_\theta = \frac{6}{H^2}M_1.$$

When the thickness of the plate varies linearly, we have $m = 3$, and for $\nu = \frac{1}{3}$ (6.74) becomes

$$\left.\begin{aligned}
\sigma_r &= \frac{6M_1}{H^2}\cdot\frac{(8 + 3\delta)\eta^3 - 8(1 - 3\delta)\left(\frac{R_1}{r}\right)^3}{8(\eta^3 - 1) + 3(\eta^3 + 8)\delta}, \\
\sigma_\theta &= \frac{6M_1}{H^2}\cdot\frac{3(8 + 3\delta)\eta^3}{8(\eta^3 - 1) + 3\delta(\eta^3 + 8)}.
\end{aligned}\right\} \tag{6.76}$$

The following conditions are fulfilled in this case:

$$\left.\begin{array}{l} \dfrac{\partial \sigma_r}{\partial \delta} > 0, \quad \dfrac{\partial \sigma_\theta}{\partial \delta} < 0, \quad \dfrac{\partial \sigma_\theta}{\partial r} = 0 \\ \sigma_r > 0 \quad \text{for} \quad \delta = 0 \\ \sigma_\theta > 0 \quad \text{for} \quad \delta \to \infty \\ \dfrac{\partial \sigma_r}{\partial r} < 0 \quad \text{for} \quad \delta > \dfrac{1}{3} \\ \sigma_r = \sigma_\theta \quad \text{for} \quad r = R \text{ and } \delta = 1\dfrac{1}{3} \end{array}\right\}. \tag{6.77}$$

Hence, the optimum value of δ, for which the calculated stresses in the plate are minimum, is

$$\delta^{\text{opt}} = 1\frac{1}{3}. \tag{6.78}$$

We obtain from (6.76) for $\delta = \delta^{\text{opt}}$

$$\sigma_{\max}^{\text{opt}} = \frac{3\eta^3}{\eta^3 + 2} \cdot \frac{6M_1}{H^2}.$$

TABLE 26

δ	η				
	1.25	1.5	2	3	5
0	6.148	4.263	3.429	3.115	3.024
$1\frac{1}{3}$	1.482	1.884	2.400	2.793	2.953
∞	1.766	2.670	4.500	6.943	8.459

Table 26 gives values of $\sigma_{\max}\sqrt{\frac{6M_1}{H^2}}$ for some values $\eta = \frac{R_1}{R}$ and $\delta = 0$ (perfectly flexible rib), $\delta = 1\frac{1}{3}$ (optimum rib), and $\delta = \infty$ (perfectly rigid rib).

We also conclude from (6.77) that when $\delta > 1\frac{1}{3}$, the strength of the plate is determined by the stresses σ_r at $r = R$, and when $\delta < 1\frac{1}{3}$, by the stresses σ_θ along the same circle.

Annular plate rigidly clamped at the outer edge and reinforced by a rib at the inner edge

Let the rigidity of the plate be given by (6.61). Setting $\delta_1 = \infty$ in (6.64), we obtain for an axisymmetrical load:

$$\left.\begin{aligned} C_1 &= \frac{1}{\Delta_3}[R_1^{-\beta}(\vartheta_0)_{r=R}\delta - (\delta - \nu + \beta)R^{-\beta}(\vartheta_0)_{r=R_1}] \\ C_2 &= \frac{1}{\Delta_3}[(\delta - \nu - \gamma)R^{\gamma}(\vartheta_0)_{r=R_1} - R_1^{\gamma}(\vartheta_0)_{r=R}\delta] \end{aligned}\right\}, \tag{6.79}$$

where

$$\Delta_3 = (\delta - \nu + \beta)R_1^{\gamma}R^{-\beta} - (\delta - \nu - \gamma)R_1^{-\beta}R^{\gamma} \text{ [and } \delta = \delta_1'\text{]}.$$

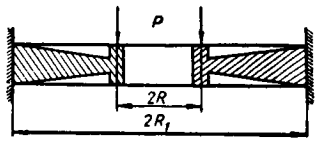


FIGURE 88.

In the particular case of a transverse load P applied to the inner edge of the plate (Figure 88), we have, similarly to (6.66),

$$\vartheta_0 = C_1^* r^{\nu} + C_2^* r^{-\beta} + \frac{P}{2\pi D_0 r^{m-1}(1-\nu)m}, \tag{6.79a}$$

where

$$C_1^* = \frac{PR_1^{1-\beta}[\eta^{2\alpha}(2\nu - 2\alpha - m) + 2\eta^{\beta-1}(m-1-\nu)]}{4\pi D_0 m(1-\nu)[(\gamma+\nu) + (\beta-\nu)\eta^{2\alpha}]},$$

$$C_2^* = \frac{PR_1^{1+\nu}[(m - 2\nu - 2\alpha) - 2\eta^{\beta-1}(m-1-\nu)]}{4\pi D_0 m(1-\nu)[(\gamma+\nu) + (\beta-\nu)\eta^{2\alpha}]}.$$

Substituting (6.79) in (6.51) and then ϑ_1 and (6,79a) in (6.50), we obtain for the case considered, the function ϑ in the form (6.69),

$$\left.\begin{aligned} C_0 &= \frac{PR_1^{1-\beta}}{\Delta_3}[\eta^{2\alpha}(2\nu - 2\delta - 2\alpha - m) + 2\eta^{\beta-1}(m - 1 - \nu + \delta)] \\ C_0^* &= \frac{PR_1^{1+\nu}}{\Delta_3}[(2\delta - 2\nu - 2\alpha + m) - 2\eta^{\beta-1}(m - 1 - \nu + \delta)] \end{aligned}\right\}, \tag{6.80}$$

where

$$\Delta_3 = 4\pi D_0 m(1-\nu)[(\gamma+\nu) + (\beta-\nu)\eta^{2\alpha} + (\eta^{2\alpha} - 1)\delta]. \tag{6.81}$$

Setting $\delta = \infty$, we obtain from (6.80) for a plate rigidly clamped also at its inner edge:

$$C_0 = \frac{PR_1^{1-\beta}}{2\pi D_0 m(1-\nu)} \cdot \frac{\eta^{\beta-1} - \eta^{2\alpha}}{\eta^{2\alpha} - 1}, \quad C_0^* = \frac{PR_1^{1+\nu}}{2\pi D_0 m(1-\nu)} \cdot \frac{1 - \eta^{\beta-1}}{\eta^{2\alpha} - 1}.$$

Assuming the plate thickness h to vary linearly ($m = 3$), and setting $\nu = \frac{1}{3}$, we obtain

$$\alpha = \frac{3}{2}, \quad \beta = 3, \quad \gamma = 0, \quad h = cr, \quad D_0 = \frac{3Ec^3}{32},$$

where c is a constant.

The solution (6.69) becomes in this case

$$\theta = C_0 + C_0^* r^{-3} + \frac{8P}{3\pi Ec^3 r^2}, \tag{6.82}$$

where

$$C_0 = \frac{8P\left[(5+3\delta)\eta^3 - (8+3\delta)\eta^2\right]}{3\pi Ec^3 R_1^2\left[1+8\eta^3+3(\eta^3-1)\delta\right]},$$

$$C_0^* = -\frac{8P\left[(1-3\delta)+(5+3\delta)\eta^2\right]R_1}{3\pi Ec^3\left[1+8\eta^3+3(\eta^3-1)\delta\right]}.$$

Integrating (6.82) with respect to r, we obtain

$$w = -C_0 r + \frac{1}{2} C_0^* r^{-2} + \frac{8P}{3\pi Ec^3 r} + C_2,$$

where C_2 is a constant, determined from the condition that $w = 0$ at $r = R_1$:

$$C_2 = C_0 R_1 - \frac{1}{2} C_0^* R_1^{-2} - \frac{8P}{3\pi Ec^3 R_1}.$$

Formulas for the deflections and stresses of plates of variable thickness with reinforced edges* can be derived similarly for other loads.

§ 5. FREE OSCILLATIONS OF CIRCULAR AND ANNULAR PLATES WITH CONCENTRIC STIFFENING RIBS

We consider in this section the free oscillations of circular and annular plates reinforced by elastic ribs (/164, 168/).

The differential equation of the free oscillations of a thin plate is (/140/, p. 214)

$$c^4 \Delta\Delta w + \frac{\partial^2 w}{\partial t^2} = 0, \tag{6.83}$$

* An approximate method is given in /18/ for the strength analysis of a circular plate of variable thickness, connected to a concentric annular elastic element and loaded axisymmetrically by applying bending moments and forces in its middle plane.

where $c^4 = Eh^2/12(1-\nu^2)\mu$; Δ denotes the Laplacian operator; E is the modulus of elasticity; ν is Poisson's ratio, h is the thickness, μ the density, and w the deflections of the plate; t denotes time.

In polar coordinates (r, θ) (6.83) becomes

$$w(r, \theta, t) = W(r, \theta)\cos(pt + \varphi_0), \tag{6.84}$$

where p is the circular frequency and $\varphi_0 = \text{const}$. We then obtain:

1) for a circular plate (domain S_1)

$$W_1(r, \theta) = [A_{1n}J_n(k_1r) + C_{1n}I_n(k_1r)]\cos n\theta; \tag{6.85}$$

2) for an annular plate (domain S_0)

$$W_0(r, \theta) = [A_{0n}J_n(k_0r) + B_{0n}N_n(k_0r) + C_{0n}I_n(k_0r) + D_{0n}K_n(k_0r)]\cos n\theta. \tag{6.86}$$

Here $I_n(k_ir)$, $K_n(k_0r)$ and $J_n(k_ir)$, $N_n(k_0r)$ $(i = 0, 1)$ are Bessel functions whose arguments are, respectively, imaginary and real, $k_i^4 = p^2/c_i^4 = p^2h_i\mu_i/D_i$; μ_i is the density, h_i, the thickness, and D_i, the rigidity of the plate in domain $S_i (i = 0, 1)$; A_{in}, C_{in}, B_{0n} and D_{0n} are constants.

We shall consider some problems concerning oscillations of circular and annular plates.

Axisymmetrical oscillations of a hinged circular plate with supporting rib of finite mass

The deflections of the plate are given by (6.84), where $W(r, \theta)$ is defined by (6.85), with $n = 0$:

$$W_1(r) = A_{10}J_0(k_1r) + C_{10}I_0(k_1r). \tag{6.87}$$

The function $W(r,\theta)$ must satisfy the following boundary conditions on the support contour $r = R$:

$$(\delta_1 - \alpha_1x_1^4 + \nu)\frac{d}{dr}W_1 + R\frac{d^2}{dr^2}W_1 = 0, \quad W_1 = 0. \tag{6.88}$$

The first condition is obtained from (2.372) for $\lambda_k = 0$.

Substituting (6.87) in (6.88), and equating to zero the determinant of the system of equations obtained, we find the characteristic equation of the problem:

$$2x_1J_0(x_1)I_0(x_1) + (b_2^0 - \alpha_1x_1^4)[J_1(x_1)I_0(x_1) + I_1(x_1)J_0(x_1)] = 0, \tag{6.89}$$

where

$$b_2^0 = \delta_1 + \nu - 1, \quad \delta_1 = \frac{A}{RD_1}, \quad x_1 = k_1R.$$

TABLE 27

α_1	δ_1									
	0	0.5	1	2	3	4	5	10	20	∞
0.2	1.520	1.647	1.750	1.911	2.038	2.143	2.232	2.549	2.871	3.200
0.04	1.960	2.109	2.227	2.399	2.523	2.617	2.690	2.895	3.037	3.200
0.0008	2.215	2.355	2.457	2.599	2.694	2.763	2.815	2.957	3.059	3.200

Table 27 gives the first roots x_1 of (6.89) for various relative flexural rigidities δ_1 of the rib and for three values of α_1 ((2.374)).

The data of Table 27 were found by the method of sections. The absolute error in no case exceeds 0.001. The frequency of the free oscillations of the plate is

$$f = \frac{p}{2\pi} = \frac{x_1^2}{2\pi R^2}\sqrt{\frac{Eh_1^2}{12\mu_1(1-\nu^2)}}. \qquad (6.90)$$

Consider as example a steel plate whose dimensions are $R = 140$ mm, $h_1 = 6$ mm, $E_1 = 2 \cdot 10^6$ dynes/cm^2, $\nu = 0.3$, $\mu_1 = 7.86$ g/cm^3.

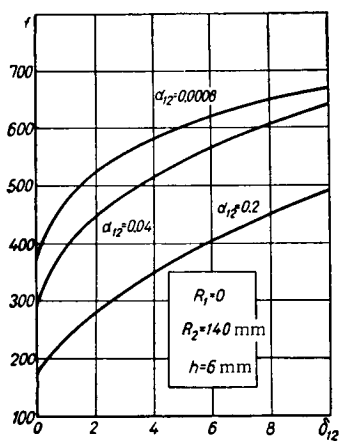


FIGURE 89.

Figure 89 shows the frequency f ((6.90)) as function of δ_1 and α_1 (i.e., of the rigidity and mass of the rib).

It is seen from the curves that f decreases with increasing mass and decreasing rigidity of the rib. When $\delta_1 \to \infty$, f tends to the value corresponding to a rigidly clamped plate. The values $\delta_1 = 0$ and $\alpha_1 = 0$ correspond to a nonreinforced plate hinged at the edge.

Experimental frequency determination for a reinforced circular plate

The above theoretical solution was tested experimentally on a circular steel plate (St. 3) having a stiffening rib of rectangular cross section (h·b).

The dimensions of plate and rib were: $R = 140$ mm, $h_1 = 6$ mm, $h = 18$ mm, $b = 5$ mm. A heavy pipe with cone-shaped upper part was used as support. The support radius was equal to the mean radius of the stiffening rib. Four resistance strain gages were symmetrically glued to the plate. The bridge thus formed was connected via an amplifier to a B-5 loop oscillograph whose natural frequency was 2500 cycles/sec.

The plate was caused to vibrate through the sudden removal, by means of a special device, of a centrally applied load. The oscillographs had also time markings at a standard frequency of 500 cycles/sec. The frequency f was found to vary between 461 and 470 cycles/sec, its mean value being 465 cycles/sec (calculated from six measurements). One such recording is shown in Figure 90.

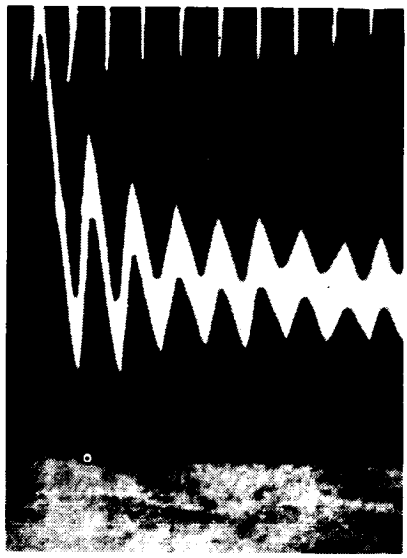


FIGURE 90.

The frequency calculated by (6.90) for the given plate with rib was 454 cycles/sec. The difference between the theoretical and experimental results is thus only 2.4%.

Free oscillations of an annular plate with edges reinforced by ribs of finite mass

An annular plate with reinforced edges $r = R$ and $r = R_1 > R$ is simply supported along the inner contour. The stiffening ribs have, respectively, rigidities A, C and A', C', densities μ and μ', moments of inertia I_t and I_t', and cross-sectional areas F and F'.

The function $W_0(r, \theta)$ given by (6.86) must, in this case, satisfy the following boundary conditions:

On the circle $r = R$, we have, according to (2.376), (2.372), and (2.375) for $D_k = 0$ and $h_k = 0$:

$$\left.\begin{aligned} -\delta_2^* \frac{\partial^3}{\partial r \partial \theta^2} W_0 + (\delta_1^* - \alpha_{11} x_2^4 - \nu_0) \frac{\partial}{\partial r} W_0 - R \frac{\partial^2}{\partial r^2} W_0 = 0 \\ W_0 = 0 \end{aligned}\right\}. \tag{6.91}$$

Replacing the subscript k of W, μ, h by 0, we obtain according to (2.372) and (2.373) on the circle $r = R_1$ for $\lambda_k = 0$:

$$\begin{aligned} &-\delta_2' \frac{\partial^3}{\partial r \partial \theta^2} W_0 + (\delta_2' + \delta_1' + \nu_0) \frac{1}{R_1} \frac{\partial^2}{\partial \theta^2} W_0 + (\delta_1' - \alpha_{12} x_2^4 + \nu_0) \times \\ &\qquad \times \frac{\partial}{\partial r} W_0 + R_1 \frac{\partial^2}{\partial r^2} W_0 = 0, \\ &-\delta_1' \frac{\partial^4}{\partial \theta^4} W_0 + (2 - \nu_0 - \delta_1' - \delta_2') R_1 \frac{\partial^3}{\partial r \partial \theta^2} W_0 + (\nu_0 - 3 + \delta_2') \times \\ &\times \frac{\partial^2}{\partial \theta^2} W_0 + R_1^3 \frac{\partial^3}{\partial r^3} W_0 + R_1^2 \frac{\partial^2}{\partial r^2} W_0 - R_1 \frac{\partial}{\partial r} W_0 + \alpha_{22} x_2^4 W_0 = 0, \end{aligned} \tag{6.92}$$

where

$$\left.\begin{aligned} \delta_1^* = \frac{A}{RD_0}, \quad \delta_2^* = \frac{C}{RD_0}, \quad \delta_1' = \frac{A'}{R_1 D_0}, \quad \delta_2' = \frac{C'}{R_1 D_0}, \\ \alpha_{11} = \mu J_1 R / \mu_0 h_0 R_1^4, \quad \alpha_{12} = \mu' J_1' / \mu_0 h_0 R_1^3, \\ \alpha_{22} = \frac{\mu' F'}{\mu_0 h_0 R_1}, \quad x_2 = k_0 R_1, \quad k_0^4 = \frac{12 p^2 \mu_0 (1 - \nu^2)}{E_0 h_0^2}, \end{aligned}\right\}, \tag{6.93}$$

h_0 is the thickness, E_0 the modulus of elasticity, μ_0 the density, and D_0 the rigidity of the plate. The boundary conditions (6.91) and (6.92) yield the following characteristic equation for the frequencies:

$$\begin{vmatrix} J_n(x_2\eta^{-1}) & N_n(x_2\eta^{-1}) & I_n(x_2\eta^{-1}) & K_n(x_2\eta^{-1}) \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0, \tag{6.94}$$

where

$$\left.\begin{aligned} a_{21} &= \frac{x_2^2}{\eta^2} J_n\left(\frac{x_2}{\eta}\right) - \frac{x_2}{\eta}\left(b_1 - \alpha_{11} x_2^4\right) J_{n+1}\left(\frac{x_2}{\eta}\right) \\ a_{22} &= \frac{x_2^2}{\eta^2} N_n\left(\frac{x_2}{\eta}\right) - \frac{x_2}{\eta}\left(b_1 - \alpha_{11} x_2^4\right) N_{n+1}\left(\frac{x_2}{\eta}\right) \\ a_{23} &= -\frac{x_2^2}{\eta^2} I_n\left(\frac{x_2}{\eta}\right) + \frac{x_2}{\eta}\left(b_1 - \alpha_{11} x_2^4\right) I_{n+1}\left(\frac{x_2}{\eta}\right) \\ a_{24} &= -\frac{x_2^2}{\eta^2} K_n\left(\frac{x_2}{\eta}\right) - \frac{x^2}{\eta}\left(b_1 - \alpha_{11} x_2^4\right) K_{n+1}\left(\frac{x^2}{\eta}\right) \\ a_{31} &= (a_2 - x_2^2 - n\alpha_{12} x_2^4) J_n(x_2) - x_2 (b_2 - \alpha_{12} x_2^4) J_{n+1}(x_2) \\ a_{32} &= (a_2 - x_2^2 - n\alpha_{12} x_2^4) N_n(x_2) - x_2 (b_2 - \alpha_{12} x_2^4) N_{n+1}(x_2) \\ a_{33} &= (a_2 + x_2^2 - n\alpha_{12} x_2^4) I_n(x_2) + x_2 (b_2 - \alpha_{12} x_2^4) I_{n+1}(x_2) \end{aligned}\right\}. \tag{6.95}$$

$$
\left.
\begin{aligned}
a_{34} &= (a_2 + x_2^2 - n\alpha_{12}x_2^4)\,K_n(x_2) - x_2(b_2 - \alpha_{12}x_2^4)\,K_{n+1}(x_2)\\
a_{41} &= (c_2 + nx_2^2 - \alpha_{22}x_2^4)\,J_n(x_2) + x_2(d_2 - x_2^2)\,J_{n+1}(x_2)\\
a_{42} &= (c_2 + nx_2^2 - \alpha_{22}x_2^4)\,N_n(x_2) + x_2(d_2 - x_2^2)\,N_{n+1}(x_2)\\
a_{43} &= (c_2 - nx_2^2 - \alpha_{22}x_2^4)\,I_n(x_2) - x_2(d_2 + x_2^2)\,I_{n+1}(x_2)\\
a_{44} &= (c_2 - nx_2^2 - \alpha_{22}x_2^4)\,K_n(x_2) + x_2(d_2 + x_2^2)\,K_{n+1}(x_2)\\
a_2 &= n^2\delta_2' - n^2(\delta_2' + \delta_1' + \nu) + n(\delta_1' + \nu) + n(n-1)\\
b_1 &= n^2\delta_2^* + \delta_1^* + 1 - \nu\\
b_2 &= n^2\delta_2' + \delta_1' - 1 + \nu\\
c_2 &= n^4\delta_1' - n^3(\delta_1' + \delta_2' + \nu) + n^2(\delta_2' + \nu) + n^2(n-1)\\
d_2 &= n^2(\delta_1' + \delta_2' - 1 + \nu)\\
\eta &= \frac{R_1}{R}
\end{aligned}
\right\} \qquad (6.95a)
$$

Using (6.89) or (6.94), we can also solve the inverse problem, i. e., determine the rigidities of the stiffening ribs, required for a given frequency of free oscillations of the plate. Clearly, this problem has no unique solution. The specified value of f must lie within the range of frequencies for nonreinforced hinged and rigidly clamped plates.

Free oscillations of circular and annular plates with stiffening ribs of zero mass

It is assumed that the mass of the ribs is so small that it can be neglected.

We consider a circular plate of radius R_1, reinforced by an elastic rib along the circle $r = R_1$ and by another concentric rib of radius $R < R_1$. These ribs have uniform cross sections, one of whose principal axes of inertia lies in the middle plane of the plate.

The axial line of the inner rib divides the domain of the plate into two parts $S_1 (r \leqslant R)$ and $S_0 (R \leqslant r \leqslant R_1)$. The corresponding flexural rigidities are $D_i = E_i h_i^3 / 12(1 - \nu_i^2)$ $(i = 0, 1)$.

The deflections of the plate in domains S_i are represented in the form (6.84):

$$
w_i(r, \theta, t) = W_i(r, \theta)\cos(pt + \varphi_0), \qquad (6.96)
$$

where the functions $W_i (i = 0, 1)$ are defined by (6.85) and (6.86).

If the mass of the ribs is neglected as being small compared with the mass of the plate, we can determine the unknown coefficients A_{in}, B_{2n}, C_{in}, $D_{2n} (i = 0, 1)$ by using the boundary conditions (2.357) at the joint between plate and inner rib, where $k = 1$, $A_k = A$, $C_k = C$, $\lambda_k = \lambda$, and (2.359), where $m_0(\theta) = p_0(\theta) = 0$, $k = 0$, $\delta_1 = \delta_1' = \dfrac{A'}{R_1 D_0}$, $\delta_2 = \delta_2'$, or (2.364), with $w = w_0$, $\delta_1 = \delta_1'$, $m_0(\theta) = 0$, depending upon whether the outer reinforced edge $r = R_1$ of the plate is free or hinged. These cases will be considered separately.

Characteristic equation for a free plate (Figure 91,a). Substituting (6.96) in (2.357) and (2.359), and equating to zero the

determinant of the system of equations in the unknown coefficients, we obtain the following characteristic equation:

$$\begin{vmatrix} s_{11}(x_1) & s_{12}(x_1) & \lambda s_{13}(x_0) & \lambda s_{14}(x_0) & \lambda s_{15}(x_0) & \lambda s_{16}(x_0) \\ s_{21}(x_1) & s_{22}(x_1) & \lambda s_{23}(x_0) & \lambda s_{24}(x_0) & \lambda s_{25}(x_0) & \lambda s_{26}(x_0) \\ -J_n(x_1) & -I_n(x_1) & J_n(x_0) & N_n(x_0) & I_n(x_0) & K_n(x_0) \\ \frac{J_{n+1}(x_1)}{x_1^{-1}} & \frac{I_{n+1}(x_1)}{x_1^{-1}} & \frac{I_{n+1}(x_0)}{x_0^{-1}} & \frac{N_{n+1}(x_0)}{x_0^{-1}} & -\frac{I_{n+1}(x_0)}{x_0^{-1}} & \frac{K_{n+1}(x_0)}{x_0^{-1}} \\ 0 & 0 & s_{53}(x_0\eta) & s_{54}(x_0\eta) & s_{55}(x_0\eta) & s_{56}(x_0\eta) \\ 0 & 0 & s_{63}(x_0\eta) & s_{64}(x_0\eta) & s_{65}(x_0\eta) & s_{66}(x_0\eta) \end{vmatrix} = 0, \tag{6.97}$$

where

$$\left.\begin{aligned} s_{11}(x_1) &= -(x_1^2 + a) J_n(x_1) - b x_1 J_{n+1}(x_1) \\ s_{12}(x_1) &= (x_1^2 - a) I_n(x_1) + b x_1 I_{n+1}(x_1) \\ s_{13}(x_0) &= [x_0^2 - n(n-1)] J_n(x_0) - x_0 J_{n+1}(x_0) \\ s_{14}(x_0) &= [x_0^2 - n(n-1)] N_n(x_0) - x_0 N_{n+1}(x_0) \end{aligned}\right\},$$

$$\left.\begin{aligned} s_{15}(x_0) &= -[x_0^2 + n(n-1)] I_n(x_0) + x_0 I_{n+1}(x_0) \\ s_{16}(x_0) &= -[x_0^2 + n(n-1)] K_n(x_0) - x_0 K_{n+1}(x_0) \\ s_{21}(x_1) &= c + n x_1^2) J_n(x_1) + (x_1 d - x_1^3) J_{n+1}(x_1) \\ s_{22}(x_1) &= (c - n x_1^2) I_n(x_1) - x_1 (d + x_1^2) I_{n+1}(x_1) \\ s_{23}(x_0) &= [(n-1)^2 - x_0^2] n J_n(x_0) + [x_0^2 - (n^2+1)] x_0 J_{n+1}(x_0) \\ s_{24}(x_0) &= [(n-1)^2 - x_0^2] n N_n(x_0) + [x_0^2 - (n^2+1)] x_0 N_{n+1}(x_0) \\ s_{25}(x_0) &= [(n-1)^2 + x_0^2] n I_n(x_0) + (x_0^2 + n^2 + 1) x_0 I_{n+1}(x_0) \\ s_{26}(x_0) &= [(n-1)^2 + x_0^2] n K_n(x_0) - (x_0^2 + n^2 + 1) x_0 K_{n+1}(x_0) \\ s_{53}(x_0\eta) &= -(x_0^2\eta^2 + a') J_n(x_0\eta) - b' x_0\eta J_{n+1}(x_0\eta) \\ s_{54}(x_0\eta) &= -(x_0^2\eta^2 + a') N_n(x_0\eta) - b' x_0\eta N_{n+1}(x_0\eta) \\ s_{55}(x_0\eta) &= (x_0^2\eta^2 - a') I_n(x_0\eta) + b' x_0\eta I_{n+1}(x_0\eta) \\ s_{56}(x_0\eta) &= (x_0^2\eta^2 - a') K_n(x_0\eta) - b' x_0\eta K_{n+1}(x_0\eta) \\ s_{63}(x_0\eta) &= (c' + n x_0^2\eta^2) J_n(x_0\eta) + (d' - x_0^2\eta^2) J_{n+1}(x_0\eta) x_0\eta \\ s_{64}(x_0\eta) &= (c' + n x_0^2\eta^2) N_n(x_0\eta) + (d' - x_0^2\eta^2) x_0\eta N_{n+1}(x_0\eta) \\ s_{65}(x_0\eta) &= (c' - n x_0^2\eta^2) I_n(x_0\eta) - (d' + x_0^2\eta^2) x_0\eta I_{n+1}(x_0\eta) \\ s_{66}(x_0\eta) &= (c' - n x_0^2\eta^2) K_n(x_0\eta) + (d' + x_0^2\eta^2) x_0\eta K_{n+1}(x_0\eta) \\ a &= n^2(\delta_1 + \delta_2 + \nu_1 - 1 - \lambda\nu_0) - n(\delta_1 + \nu_1 - \lambda\nu_0 + \delta_2 n^2 - 1) \\ b &= \delta_1 + \nu_1 - \lambda\nu_0 + \delta_2 n^2 - 1 \\ c &= n^4\delta_1 + n^3(\delta_2 + \nu_1 - 1 - \lambda\nu_0 + 3\lambda) - n^3(\delta_1 + \delta_2 + \nu_1 - \\ &\quad - 1 - \lambda\nu_0 + 2\lambda) - \lambda n \\ d &= n^2(\delta_1 + \delta_2 + \nu_1 - 1 - \lambda\nu_0 + 2\lambda) + \lambda \\ a' &= n^2(\delta_1' + \delta_2' + \nu_0 - 1) - n(\delta_1' + \nu_0 + \delta_2' n^2 - 1) \end{aligned}\right\}, \tag{6.98}$$

$$
\begin{aligned}
b' &= \delta_1' + \nu_0 + \delta_2' n^2 - 1 \\
c' &= n^4\delta_1' + n^2(\delta_2' + \nu_0 - 1) - n^3(\delta_1' + \delta_2' + \nu_0 - 1) \\
d' &= n^2(\delta_1' + \delta_2' + \nu_0 - 1)
\end{aligned}
\qquad (6.98)\ \text{cont'd}
$$

$$
x_1 = k_1 R, \quad x_0 = k_0 R, \quad \eta = \frac{R_1}{R}, \quad \lambda = \frac{D_0}{D_1},
$$

$$
\delta_1 = \frac{A}{RD_1}, \quad \delta_2 = \frac{C}{RD_1}, \quad \delta_1' = \frac{A'}{R_1 D_0}, \quad \delta_2' = \frac{C'}{R_1 D_0}, \qquad (6.99)
$$

A, C and A', C' are, respectively, the flexural and torsional rigidities of the inner and outer ribs.

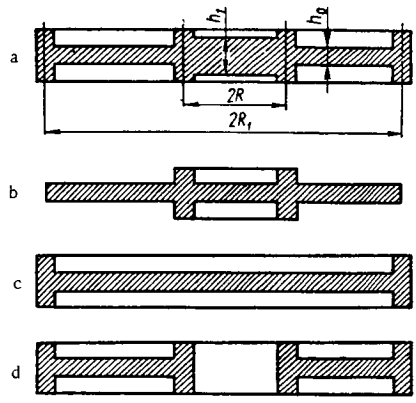


FIGURE 91.

The oscillation frequency is

$$
f = \frac{x_1^2\eta^2}{2\pi R_1^2}\sqrt{\frac{D_1}{\mu_1 h_1}} = \frac{x_0^2\eta^2}{2\pi R_1^2}\sqrt{\frac{D_0}{\mu_0 h_0}}. \qquad (6.100)
$$

We shall consider some particular cases.

a) Setting $\delta_1 = \delta_2 = 0$, $\lambda = 1$, $k_1 = k_0$, and $n = 0$, we obtain from (6.97) the characteristic equation for axisymmetrical oscillations of a homogeneous circular plate of uniform thickness, reinforced by a concentric stiffening rib along the circle $r = R$ (Figure 91,b):

$$
\begin{aligned}
&\frac{2}{x_1\eta} J_1(x_1) I_1(x_1) + [N_0^*(x_1\eta) + BK_1(x_1\eta) J_1(x_1\eta)] I_1^2(x_1) + \\
&\quad + \frac{\pi}{2}[K_0^*(x_1\eta) - BN_1(x_1\eta) I_1(x_1\eta)] J_1^2(x_1) + \\
&+ [M_0^*(x_1\eta) - BJ_1(x_1\eta) I_1(x_1\eta)] \cdot \left[\frac{2}{\delta_1} + K_1(x_1) I_1(x_1) - \right. \\
&\quad \left. - \frac{\pi}{2} J_1(x_1) N_1(x_1)\right] = 0,
\end{aligned}
\qquad (6.101)
$$

where

$$B = 2(1-\nu_1)/x_1\eta, \quad N_0^*(x_1\eta) = K_0(x_1\eta)J_1(x_1\eta) - J_0(x_1\eta)K_1(x_1\eta),$$
$$M_0^*(x_1\eta) = J_0(x_1\eta)I_1(x_1\eta) + J_1(x_1\eta)I_0(x_1\eta), \tag{6.102}$$
$$K_0^*(x_1\eta) = N_0(x_1\eta)I_1(x_1\eta) + N_1(x_1\eta)I_0(x_1\eta).$$

Setting $\eta = 1$ in (6.101), we have the case of a stiffening rib located at the plate edge (Figure 91,c):

$$M_0^*(x_1) + 2(\delta_1 + \nu_1 - 1)x_1^{-1}J_1(x_1)I_1(x_1) = 0. \tag{6.103}$$

b) For $\lambda = \infty$, we obtain the characteristic equation for a free annular plate whose edges are reinforced by thin elastic ribs (Figure 91,d):

$$\begin{vmatrix} s_{11}^*(x_0) & s_{12}^*(x_0) & s_{13}^*(x_0) & s_{14}^*(x_0) \\ s_{21}^*(x_0) & s_{22}^*(x_0) & s_{23}^*(x_0) & s_{24}^*(x_0) \\ s_{53}(x_0\eta) & s_{54}(x_0\eta) & s_{55}(x_0\eta) & s_{56}(x_0\eta) \\ s_{63}(x_0\eta) & s_{64}(x_0\eta) & s_{65}(x_0\eta) & s_{66}(x_0\eta) \end{vmatrix} = 0, \tag{6.104}$$

where

$$\left.\begin{aligned}
s_{11}^*(x_0) &= (x_0^2 - a^*)J_n(x_0) - b^*x_0J_{n+1}(x_0), \\
s_{12}^*(x_0) &= (x_0^2 - a^*)N_n(x_0) - b^*x_0N_{n+1}(x_0), \\
s_{13}^*(x_0) &= -(x_0^2 + a^*)I_n(x_0) + b^*x_0I_{n+1}(x_0), \\
s_{14}^*(x_0) &= -(x_0^2 + a^*)K_n(x_0) - b^*x_0K_{n+1}(x_0), \\
s_{21}^*(x_0) &= (c^* - nx_0^2)J_n(x_0) + (d^* + x_0^2)x_0J_{n+1}(x_0), \\
s_{22}^*(x_0) &= (c^* - nx_0^2)N_n(x_0) + (d^* + x_0^2)x_0N_{n+1}(x_0), \\
s_{23}^*(x_0) &= (c^* + nx_0^2)I_n(x_0) - (d^* - x_0^2)x_0J_{n+1}(x_0), \\
s_{24}^*(x_0) &= (c^* + nx_0^2)K_n(x_0) + (d^* - x_0^2)x_0K_{n+1}(x_0), \\
a^* &= n^2(\delta_1^* + \delta_2^* - \nu_0 + 1) - n(\delta_1^* - \nu_0 + \delta_2^*n^2 + 1), \\
b^* &= \delta_1^* - \nu_0 + n^2\delta_3^* + 1, \\
c^* &= n^4\delta_1^* + n^2(\delta_2^* + 1 - \nu_0) - n^3(\delta_1^* + \delta_2^* + 1 - \nu_0), \\
d^* &= n^2(\delta_1^* + \delta_2^* + 1 - \nu_0),
\end{aligned}\right\}, \tag{6.105}$$

$$\delta_1^* = \frac{A}{RD_0}, \quad \delta_2^* = \frac{C}{RD_0}.$$

Characteristic equation for a supported plate. The characteristic equation for a plate supported along the circle $r = R_1$ (Figure 92) is obtained by using (2.384) instead of (2.359). This is equivalent to replacing the last two rows of (6.97) by the following:

$$\begin{vmatrix} 0 & 0 & J_n(x_0\eta) & N_n(x_0\eta) & I_n(x_0\eta) & K_n(x_0\eta) \\ 0 & 0 & -J_{n+1}(x_0\eta) & -N_{n+1}(x_0\eta) & s_1 & s_2 \end{vmatrix},$$

where

$$s_2=-K_{n+1}(x_0\eta)+\frac{2x_0\eta}{b'}K_n(x_0\eta),\qquad s_1=I_{n+1}(x_0\eta)+\frac{2x_0\eta}{b'}I_n(x_0\eta).$$

We shall consider some particular cases.

a) For $\delta_1'=\delta_2'=0$, $\lambda=1$, $\nu_1=\nu_2=\nu$, $k_1=k_0$, and $n=0$ we obtain the characteristic equation corresponding to axisymmetrical oscillations of a homogeneous circular plate of uniform thickness with one concentric stiffening at $r=R$ (Figure 92,b):

$$\begin{aligned}&-\frac{2}{x_1\eta}J_1(x_1)I_1(x_1)+[N_0^*(x_1\eta)-4B^{-1}J_0(x_1\eta)K_0(x_1\eta)]I_1^2(x_1)+\\&+\frac{\pi}{2}[K_0^*(x_1\eta)-4B^{-1}N_0(x_1\eta)I_0(x_1\eta)]J_1^2(x_1)+\\&+\left[\frac{2}{\delta_1}+K_1(x_1)I_1(x_1)-\frac{\pi}{2}J_1(x_1)N_1(x_1)\right]\cdot[M_0^*(x_1\eta)-\\&-4B^{-1}I_0(x_1\eta)J_0(x_1\eta)]=0.\end{aligned}\tag{6.106}$$

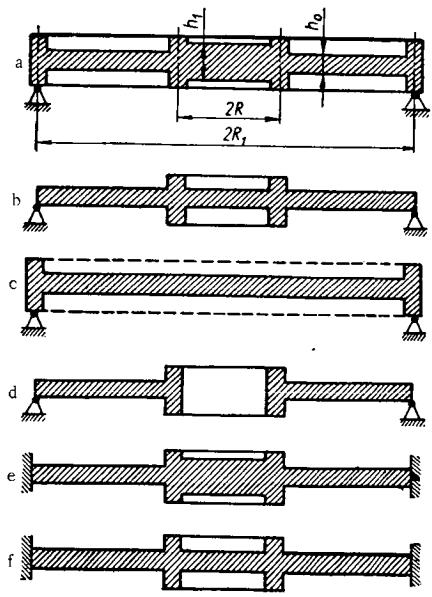


FIGURE 92.

b) Setting $\delta_1'=\delta_2'=0$ and $\eta=1$, we obtain the characteristic equation for a homogeneous circular plate reinforced by a rib at the edge $r=R_1$ (Figure 92,c)

$$2x_1I_n(x_1)J_n(x_1)+(\delta_1+n^2\delta_2-1+\nu)\times M_n^*(x_1)=0,\tag{6.107}$$

where

$$M_n^*(x_1) = J_n(x_1) I_{n+1}(x_1) + J_{n+1}(x_1) I_n(x_1).$$

c) For $\lambda = \infty, \delta_1' = \delta_2' = 0$, we obtain the characteristic equation for a hinged circular plate with central hole whose edge is reinforced by a thin rib (Figure 92,d). This equation can also be obtained by replacing the last two rows in (6.104) by

$$\begin{vmatrix} J_n(x_0\eta) & N_n(x_0\eta) & I_n(x_0\eta) & K_n(x_0\eta) \\ -J_{n+1}(x_0\eta) & -N_{n+1}(x_0\eta) & s' & s'' \end{vmatrix},$$

where

$$s' = I_{n+1}(x_0\eta) - \frac{2x_0\eta}{1-\nu_0}(x_0\eta), \quad s'' = -K_{n+1}(x_0\eta) - \frac{2x_0\eta}{1-\nu_0} K_n(x_0\eta).$$

d) The characteristic equation for a plate rigidly clamped along the edge $r = R_1$ ($\delta_1' = \delta_2' = \infty$) and having a thin rib of radius $R < R_1$ (cf. Figure 92,e) is obtained from (6.97) by replacing the last two rows by

$$\begin{vmatrix} 0 & 0 & J_n(x_0\eta) & N_n(x_0\eta) & I_n(x_0\eta) & K_n(x_0\eta) \\ 0 & 0 & -J_{n+1}(x_0\eta) & -N_{n+1}(x_0\eta) & I_{n+1}(x_0\eta) & -K_{n+1}(x_0\eta) \end{vmatrix}.$$

3) In the case of axisymmetrical ($n=0$) oscillations of a homogeneous circular plate ($\lambda = 1, k_1 = k_0$) clamped at the edge ($\delta_1' = \delta_2' = \infty$) and having a stiffening rib at $r = R$ (Figure 92,f), we obtain the following characteristic equation:

$$-\frac{2}{x_1\eta} J_1(x_1) I_1(x_1) + I_1^2(x_1) N_0^*(x_1\eta) + \frac{\pi}{2} J_1^2(x_1) K_0^*(x_1\eta) + \\ + M_0^*(x_1\eta)\left\{\frac{2}{\delta_1} + K_1(x_1) I_1(x_1) - \frac{\pi}{2} J_1(x_1) N_1(x_1)\right\} = 0. \qquad (6.108)$$

Influence of a concentric rib on the oscillation frequency of a built-in circular plate

As example we shall study the influence of a stiffening rib on the frequency of axisymmetrical oscillations of a circular plate clamped at the edge (Figure 92,f). Table 28 gives the roots $x_1\eta$ of (6.108) for different relative rigidities δ_1 of the stiffening rib and for different positions of the latter $\left(\eta = \frac{R_1}{R}\right)$.

Curves of $x_1^2\eta^2$ as function of $\eta^* = \frac{R}{R_1}$ (from which the oscillation frequency of the plate can be found by (6.100)) are given in Figure 93 for $\delta = \delta_1\eta^* = \frac{A}{R_1 D_0}$ which determines the flexural rigidity of the rib.

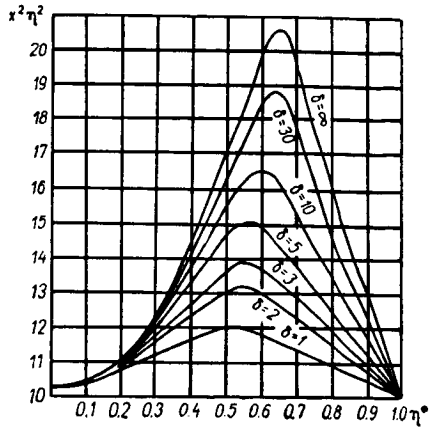


FIGURE 93.

TABLE 28

δ_1	η									
	1	1.25	1.5	1.75	2	3	4	5	10	∞
0	3.196	3.196	3.196	3.196	3.196	3.196	3.196	3.196	3.196	3.196
1	3.196	3.274	3.339	3.365	3.358	3.300	3.256	3.233	3.207	3.196
2	3.196	3.345	3.452	3.487	3.466	3.354	3.294	3.261	3.213	3.196
3	3.196	3.385	3.540	3.588	3.555	3.390	3.314	3.274	3.216	3.196
5	3.196	3.471	3.647	3.724	3.676	3.429	3.338	3.290	3.220	3.196
10	3.196	3.595	3.897	3.921	3.862	3.495	3.366	3.307	3.224	3.196
100	3.196	3.938	4.405	4.319	4.098	3.682	3.402	3.330	3.229	3.196
∞	3.196	3.954	4.497	4.396	4.138	3.600	3.409	3.333	3.230	3.196

It is seen that the oscillation frequency increases with the relative rib rigidity δ. The maximum frequency is attained for a rib radius R between 0.5 and $0.65R_1$. Let the stiffening rib have a rectangular cross section with sides b and h. Setting $\frac{b}{R} = 0.143$, $\frac{R}{R_1} = 0.55$, $\frac{h}{h_1} = 3.5$, and $\nu = 0.3$, we obtain $\delta = \left(\frac{b}{R}\right)\left(\frac{R}{R_1}\right)\left(\frac{h^3}{h_1^3} - 1\right)(1-\nu^2) = 3$. We then find from the diagram that $x_1^2\eta^2 = 14$. In this case the frequency of the reinforced plate is 37.1% higher than that of the nonreinforced plate for which $x_1^2\eta^2 = 10.21$. Since we have neglected the mass of the stiffening rib, the curves in Figure 93 define the upper limits of the frequencies.

Using boundary conditions (2.357) and (2.359) or (2.364), we can similarly derive the characteristic equation for a circular or annular plate reinforced by more than two stiffening ribs.

§ 6. AXISYMMETRICAL BENDING OF THIN PLATES WITH CONCENTRIC THIN RIBS

We shall consider a thin circular or annular plate of thickness h, using a polar system of coordinates (r, θ). The plate edge is reinforced by a thin

elastic rib considered as a thin curved bar possessing tensile rigidity and flexural rigidity in two planes. We assume that the axial line Γ of the rib lies in the middle plane of the plate, to which one of the principal axes of inertia of each cross section is perpendicular. The plate carries an axisymmetrical load $q(r)$ per unit area.

The stresses in the thin plate are in this case (/17/, p. 178) determined by a deflection function $w(r)$ and a stress function $\Phi(r)$, which must satisfy the equilibrium equation

$$D\frac{d}{dr}(\Delta w) = \psi + \frac{h}{r}\cdot\frac{d\Phi}{dr}\cdot\frac{dw}{dr} \tag{6.109}$$

and the compatibility equation

$$\frac{d}{dr}(\Delta\Phi) = -\frac{E}{2r}\left(\frac{dw}{dr}\right)^2, \tag{6.110}$$

where

$$\psi = \frac{1}{r}\int_0^r q(r)\,r\,dr. \tag{6.111}$$

The bending moments and the stresses in the middle surface of the plate, are

$$\begin{aligned} M_r &= -D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\cdot\frac{dw}{dr}\right), \quad \sigma_r = \frac{1}{r}\cdot\frac{d\Phi}{dr}, \\ M_\theta &= -D\left(\frac{1}{r}\cdot\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right), \quad \sigma_\theta = \frac{d^2\Phi}{dr^2}. \end{aligned} \tag{6.112}$$

The strains in the middle surface are

$$\varepsilon_r = \frac{1}{E}\left(\frac{1}{r}\cdot\frac{d\Phi}{dr} - \nu\frac{d^2\Phi}{dr^2}\right), \quad \varepsilon_\theta = \frac{1}{E}\left(\frac{d^2\Phi}{dr^2} - \frac{\nu}{r}\cdot\frac{d\Phi}{dr}\right). \tag{6.113}$$

The functions w and Φ must satisfy certain boundary conditions at the plate edges. The boundary conditions for the deflections w were derived in § 16 of Chapter II.

Thus, if the outer support contour $r = R$ of a circular or annular plate is reinforced by a thin elastic rib, we have for $r = R$ [cf. (2.366)]

$$(\delta_1 + \nu)\frac{dw}{dr} + R\frac{d^2w}{dr^2} = 0, \quad w = 0, \tag{6.114}$$

where $\delta_1 = EJ/RD$ is the relative flexural rigidity of the rib (J is the moment of inertia of the section with respect to the principal axis lying in the middle plane of the plate).

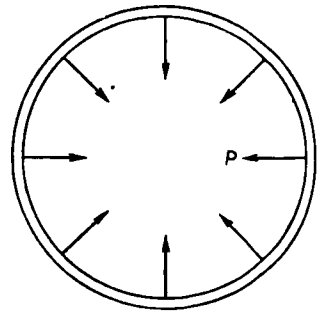


FIGURE 94.

If the inner edge $r = R$ of an annular plate is reinforced by a thin rib and subjected to a transverse load $p_0 = \text{const.}$, the deflection w satisfies the condition [at $r = R$ cf. (2.367)]

$$\left.\begin{aligned}(\delta_1 - \nu)\frac{dw}{dr} - R\frac{d^2w}{dr^2} &= 0\\ -\frac{dw}{dr} + R\frac{d^2w}{dr^2} + & \\ + R^2\frac{d^3w}{dr^3} &= p_0\frac{R^2}{D}\end{aligned}\right\}. \tag{6.115}$$

We shall now derive the boundary conditions for the function Φ. The plate acts on the rib with the radial forces $P = \text{const.}$ [per unit length] (Figure 94), in addition to the bending moments and shearing forces taken into account in (6.114) or (6.115).

We require that the following conditions be satisfied on the axial line Γ of the rib: 1) the tangential strains of plate (ε_θ) and rib (ε_*) must be equal; and 2) the radial forces of interaction between plate and rib must be equal and opposite:

$$\varepsilon_\theta = \varepsilon_*, \quad h\sigma_r = \pm P \text{ on } \Gamma. \tag{6.116}$$

Here σ_r are the radial stresses in the middle surface of the plate. The upper sign corresponds to a rib reinforcing the outer edge of a circular or annular plate, and the lower sign to a rib reinforcing the inner edge of an annular plate.

We determine the tangential strain ε_* of the rib from Hooke's law for a curved bar (/141/, Vol. 1, pp. 117-119):

$$\varepsilon_* = \frac{N}{g_1} + \frac{M}{Rg_1}, \quad \beta - \varepsilon_* = \frac{MR}{g_2}, \tag{6.117}$$

where N is the longitudinal force and M the bending moment acting on the rib section; β is the change [per radian] arc length, in the angle of rotation of the section; $g_1 = E_1F$ is the tensile rigidity of the bar; $g_2 = E_1J'$ is its flexural rigidity.

In the axisymmetrical case we have

$$\beta = 0, \quad M = \text{const}, \quad N = \text{const}. \tag{6.118}$$

Since the stiffening rib is thin, and the largest dimension of its cross section in the plane of curvature is small compared with R, the symbol J' in g_2 can be understood to mean simply the moment of inertia of the section with respect to its principal axis normal to the plane of the rib.

The longitudinal force N is in the case considered (/137/, p. 228),

$$N = -PR. \tag{6.119}$$

Substitution of (6.118) and (6.119) in (6.117) yields:

$$\varepsilon_* = -\frac{PR}{g_1 + g_2 R^{-2}}. \tag{6.120}$$

If we neglect the flexural rigidity g_2, as is usual when considering thin bars of small curvature, we would obtain

$$\varepsilon_* = -\frac{PR}{g_1}. \tag{6.121}$$

Comparison of (6.121) with (6.120) shows that we can consider the magnitude

$$B = g_1 + g_2 R^{-2} = E_1 F + E_1 J' R^{-2} \tag{6.122}$$

as reduced flexural rigidity of the rib.

Substituting (6.120) in (6.116) and eliminating P, we obtain a further boundary condition for the joint between plate and rib:

$$\varepsilon_\vartheta = \mp \frac{Rh}{B}\sigma_r \text{ at } r = R. \tag{6.123}$$

Expressing σ_r and ε_ϑ in (6.123) through the stress function $\Phi(r)$ according to (6.112) and (6.113), we obtain a third boundary condition on Γ:

$$\frac{d^2\Phi}{dr^2} + \frac{\pm 1 - \nu\delta_3}{r\delta_3} \cdot \frac{d\Phi}{dr} = 0 \text{ at } r = R \tag{6.124}$$

where

$$\delta_3 = \frac{B}{EhR} \tag{6.125}$$

determines the tensile rigidity of the rib.

For $\delta_3 = \infty$, we obtain from (6.124) the condition of immobility in the radial direction of a point on Γ:

$$\frac{d^2\Phi}{dr^2} - \frac{\nu}{r} \cdot \frac{d\Phi}{dr} = 0 \text{ at } r = R.$$

For $\delta_3 = 0$, we obtain from (6.124) the condition that the radial stresses σ_r vanish:

$$\frac{d\Phi}{dr} = 0 \text{ at } r = R.$$

In addition, the stresses σ_r must be finite at the center of a circular plate. The necessary condition for this is

$$\frac{d\Phi}{dr} = 0 \text{ at } r = 0. \tag{6.126}$$

According to (6.112), we can write (6.124) as follows:

$$r\frac{d}{dr}\sigma_r + \left(1 - \nu \pm \frac{1}{\delta_3}\right)\sigma_r = 0 \text{ at } r = R. \tag{6.127}$$

If a circular plate ($r \leqslant R_1$) is reinforced by a concentric thin circular rib of radius $R < R_1$, we obtain at $r = R$ four conditions for the function w and two conditions for the function Φ.

Equations (6.115) are then replaced by the four conditions (2.368), and (6.123) by

$$\varepsilon_{\theta 1} = \varepsilon_{\theta 0} = (h_0\sigma_{r0} - h_1\sigma_{r1})\frac{R}{B} \text{ at } r = R, \tag{6.128}$$

where the subscripts 0 and 1 refer, respectively, to the domains $r \geqslant R$ and $r \leqslant R$. We assume for the sake of generality that the plate thickness inside the rib differs from that outside.

Proceeding as before, we reduce (6.128) to the following two boundary conditions:

$$\left.\begin{aligned} \frac{d^2\Phi_1}{dr^2} - \frac{\nu_1}{r}\cdot\frac{d\Phi_1}{dr} &= -\frac{1}{R\delta_3}\left(\frac{d\Phi_1}{dr} - \frac{d\Phi_0}{dr}\cdot\frac{h_0}{h_1}\right), \\ \frac{d^2\Phi_0}{dr^2} - \frac{\nu_0}{r}\cdot\frac{d\Phi_0}{dr} &= -\frac{1}{R\delta_3}\left(\frac{d\Phi_1}{dr} - \frac{d\Phi_0}{dr}\cdot\frac{h_0}{h_1}\right), \end{aligned}\right\} \tag{6.129}$$

where $\delta_3 = B/Eh_1R$.

By analogy with (6.127) we can write (6.129) in the form

$$\left.\begin{aligned} r\frac{d}{dr}\sigma_{r1} + \left(1 - \nu_1 + \frac{1}{\delta_3}\right)\sigma_{r1} &= \frac{h_0}{h_1\delta_3}\sigma_{r0} \\ r\frac{d}{dr}\sigma_{r0} + \left(1 - \nu_0 - \frac{h_0}{h_1\delta_3}\right)\sigma_{r0} &= -\frac{1}{\delta_3}\sigma_{r1} \end{aligned}\right\}. \tag{6.130}$$

Problems of axisymmetrical bending of thin plates with reinforced edges will be solved below approximately by means of Bubnov and Galerkin's variational method and the perturbation method (method of a small parameter).

Bubnov's and Galerkin's method applied to the bending of a circular plate

We represent the deflections w of a circular plate in the form

$$w = \sum_{i=1}^{n} A_i \eta_i, \tag{6.131}$$

where the functions $\eta_i = \eta_i(r)$ satisfy the boundary conditions (6.114) and the constants A_i are determined from the variational equations (/17/, p. 185)

$$\int_0^R X \frac{d\eta_i}{dr} r dr = 0 \qquad (i = 1,2,\ldots,n), \tag{6.132}$$

$$X = D\frac{d}{dr}(\Delta w) - \psi - \frac{h}{r} \cdot \frac{d\Phi}{dr} \cdot \frac{dw}{dr}. \tag{6.133}$$

The function ψ is determined from (6.111), and the derivative $\frac{d\Phi}{dr}$ of the stress function from the compatibility equation (6.110) whose integration yields

$$\Delta\Phi = \frac{1}{r} \cdot \frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = -\frac{E}{2}\int_0^r \frac{1}{r}\left(\frac{dw}{dr}\right)^2 dr + C_1. \tag{6.134}$$

Multiplying by r and integrating again, we obtain

$$\frac{d\Phi}{dr} = -\frac{E}{2r}\int_0^r r\left[\int_0^r \frac{1}{r}\left(\frac{dw}{dr}\right)^2 dr\right] dr + \frac{C_1 r}{2} + \frac{C_2}{r}. \tag{6.135}$$

The integration constants C_1 and C_2 are determined from the boundary conditions (6.124) and (6.126) for the function Φ. From (6.126) we find that $C_2 = 0$, and from (6.124) we obtain

$$C_1 = \frac{E\delta_3}{1 + (1-\nu)\delta_3}\left\{\int_0^R \frac{1}{r}\left(\frac{dw}{dr}\right)^2 dr - \right.$$

$$\left. - \frac{(1+\nu)\delta_3 - 1}{R^2\delta_3}\int_0^R r\left[\int_0^r \frac{1}{r}\left(\frac{dw}{dr}\right)^2 dr\right] dr\right\}. \tag{6.136}$$

Thus,

$$\frac{d\Phi}{dr} = -\frac{E}{2r}\int_0^r r\left[\int_0^r \frac{1}{r}\left(\frac{dw}{dr}\right)^2 dr\right] dr + \frac{C_1 r}{2}, \tag{6.137}$$

where C_1 is given by (6.136).

Perturbation method applied to the bending of a uniformly loaded circular plate

We adopt as perturbation parameter the maximum deflection f at the center or, which is the same, the dimensionless deflection $\zeta = f/h$ (h is the plate thickness). The plate carries a load q = const.

Following Ch'ien Wei-ch'an /171/, we introduce the dimensionless parameters:

$$x = 1 - \frac{r^2}{R^2}, \quad w^* = \frac{w}{h}, \quad \sigma_r^* = \frac{\sigma_r}{E}\left(\frac{R}{h}\right)^2, \quad \sigma_\theta^* = \frac{\sigma_\theta}{E}\left(\frac{R}{h}\right)^2, \quad q^* = \frac{q}{E}\left(\frac{R}{h}\right)^4. \tag{6.138}$$

Substituting (6.111) and (6.112) in (6.109) and (6.110), we obtain

$$-\frac{1}{4}\cdot\frac{d^2}{dx^2}\left[(1-x)\frac{dw^*}{dx}\right] = \frac{3}{16}q^*(1-\nu^2) - \frac{3}{4}(1-\nu^2)\sigma_r^*\frac{dw^*}{dx}, \tag{6.139}$$

$$\frac{d^2}{dx^2}[(1-x)\sigma_r^*] = -\frac{1}{2}\left(\frac{dw^*}{dx}\right)^2. \tag{6.140}$$

The functions w^*, σ_r^* and q^* can be expanded by powers of $\zeta = \frac{f}{h}$ in their domains of convergence:

$$\begin{aligned} w^* &= w_1^*(x)\zeta + w_3^*(x)\zeta^3 + \ldots, \\ \sigma_r^* &= g_2(x)\zeta^2 + g_4(x)\zeta^4 + \ldots, \\ \frac{3}{16}q^*(1-\nu^2) &= \alpha_1\zeta + \alpha_3\zeta^3 + \ldots. \end{aligned} \tag{6.141}$$

The odd-power terms in the expansion of σ_r^* and the even-power terms in the expansions of w^* and q^* cancel out in the course of the calculations.

We solve the problem of bending of a flexible plate by the perturbation method, substituting (6.141) in (6.139) and (6.140) and equating the coefficients of equal powers of ζ. This results in ordinary differential equations containing the unknown functions w_n^* and $q_n(x)$, and also the constants α_n with successively increasing subscripts. In a first approximation, we obtain from (6.139):

$$-\frac{1}{4}\cdot\frac{d^2}{dx^2}\left[(1-x)\frac{dw_1^*}{dx}\right] = \alpha_1, \tag{6.142}$$

which is the basic equation in the theory of bending of thick plates.

The second and third approximations are

$$\frac{d^2}{dx^2}[(1-x)g_2(x)] = -\frac{1}{2}\left(\frac{dw_1^*}{dx}\right)^2, \tag{6.143}$$

$$-\frac{1}{4}\cdot\frac{d^2}{dx^2}\left[(1-x)\frac{dw_3^*}{dx}\right] = \alpha_3 - \frac{3}{4}(1-\nu^2)g_2(x)\frac{dw_1^*}{dx}. \tag{6.144}$$

Equations (6.142) to (6.144) are integrated successively under the relevant boundary conditions, remembering that $w^* = \zeta$ at $r = 0$.

The influence of a stiffening rib on the bending of a thin plate will be considered below with the aid of examples.

Uniformly loaded circular plate with reinforced edge

The problem of bending of a uniformly loaded circular plate with thin elastic support rib will be solved by Bubnov and Galerkin's method. We represent the function w ((6.131)) as the deflections (2.281) of a thick plate with support rib under the same load:

$$w = fa\left(\frac{1}{a} - 2\frac{r^2}{R^2} + b\frac{r^4}{R^4}\right) = f\eta_1, \tag{6.145}$$

where R is the plate radius, $f = w_{r=0}$,

$$a = \frac{3+\nu+\delta_1}{5+\nu+\delta_1}, \quad b = \frac{1+\nu+\delta_1}{3+\nu+\delta_1}. \tag{6.146}$$

Here f is a parameter.

Substituting (6.145) in (6.137), we obtain

$$\frac{d\Phi}{dr} = \frac{Ef^2a^2}{6R}\left[-6\frac{r^3}{R^3} + 4b\frac{r^5}{R^5} - b^2\frac{r^7}{R^7} + \frac{r}{R}\{(6-4b+b^2) + \right.$$
$$\left. + [6(3-\nu) - 4(5-\nu)b + (7-\nu)b^2]\delta_3\}\frac{1}{1+(1-\nu)\delta_3}\right]. \tag{6.147}$$

For a uniform load (q = const.), (6.111) becomes

$$\psi = \frac{qr}{2}. \tag{6.148}$$

Substituting (6.148), (6.145), and (6.147) first in (6.133) and then in (6.132), we obtain

$$k_1\zeta^3 + k_2\zeta = q^*, \tag{6.149}$$

where $q^*(\zeta)$ is given by (6.138), $\zeta = \frac{f}{h}$,

$$k_1 = -\frac{2a^3}{21(2b-3)}\left[\frac{21b^2-56b+42}{1+(1-\nu)\delta_3}\{[6(3-\nu) - 4(5-\nu)b + \right.$$
$$+ (7-\nu)b^2]\delta_3 + (6-4b+b^2)\} - 12b^4 + 84b^3 - \tag{6.150}$$
$$\left. - 252b^2 + 336b - 168\right],$$

$$k_2 = \frac{16ab}{3(1-\nu^2)}. \tag{6.151}$$

We consider certain limiting cases:

a) $\delta_3 = \infty$, i. e., the stiffening rib is inextensible; points on the contour are fixed. We obtain from (6.150)

$$k_1 = -\frac{2a^3}{21(2b-3)}\left[\left(6\frac{3-\nu}{1-\nu} - 4b\frac{5-\nu}{1-\nu} + b^2\frac{7-\nu}{1-\nu}\right)(21b^2 - 56b + 42) - 12b^4 + 84b^3 - 252b^2 + 336b - 168\right]. \quad (6.152)$$

b) $\delta_3 = 0$, i. e., points on the contour reinforced by the rib move freely. In this case we obtain

$$k_1 = -\frac{2a^3}{21(2b-3)}[9b^4 - 56b^3 + 140b^2 - 168b + 84]. \quad (6.153)$$

Setting in addition $\delta_1 = 0$, i. e., assuming that in (6.150) to (6.153)

$$a = \frac{3+\nu}{5+\nu}, \quad b = \frac{1+\nu}{3+\nu},$$

we obtain the coefficients k_1 and k_2 for a nonreinforced hinged plate (/17/, §§ 41, 42).

c) $\delta_1 = \infty$, $\delta_3 = \infty$, i. e., the plate is clamped along the contour whose points are fixed. In this case $a = b = 1$, and we obtain from (6.150) and (6.151):

$$k_1 = \frac{2(23-9\nu)}{21(1-\nu)}, \quad k_2 = \frac{16}{3(1-\nu^2)}. \quad (6.154)$$

d) $\delta_1 = \infty$, $\delta_3 = 0$, i. e., the plate is clamped along the contour whose points move freely. In this case $a = b = 1$; we obtain from (6.153):

$$k_1 = \frac{6}{7}, \quad k_2 = \frac{16}{3(1-\nu^2)}. \quad (6.155)$$

The stresses in the middle surface are found by substituting (6.147) and (6.138) in (6.112):

$$\left.\begin{aligned}\sigma_r^* &= \frac{a^2\zeta^2}{6}\left[-6\frac{r^2}{R^2} + 4b\frac{r^4}{R^4} - b^2\frac{r^6}{R^6} + \frac{1}{1+(1-\nu)\delta_3}\{6 - \right.\\ &\left. - 4b + b^2 + [6(3-\nu) - 4b(5-\nu) + b^2(7-\nu)]\delta_3\}\right]\\ \sigma_\theta^* &= \frac{a^2\zeta^2}{6}\left[-18\frac{r^2}{R^2} + 20b\frac{r^4}{R^4} - 7b^2\frac{r^6}{R^6} + \frac{1}{1+(1-\nu)\delta_3}\{6 - \right.\\ &\left. - 4b + b^2 + [6(3-\nu) - 4b(5-\nu) + b^2(7-\nu)]\delta_3\}\right]\end{aligned}\right\}. \quad (6.156)$$

At the plate center we have

$$\sigma^* = \sigma_\theta^* = \frac{a^2\zeta^2}{6[1+(1-\nu)\delta_3]}\{6-4b+b^2+ \\ +[6(3-\nu)-4(5-\nu)b+(7-\nu)b^2]\delta_3\}. \tag{6.157}$$

The maximum bending stresses are

$$\sigma_{r,u} = \frac{6M_r}{h^2}, \quad \sigma_{\theta,u} = \frac{6M_\theta}{h^2}, \tag{6.158}$$

where the bending moments M_r and M_θ (6.112) are expressed through the deflections (6.145). Writing

$$\sigma_{r,u}^* = \frac{\sigma_{r,u}}{E}\left(\frac{R}{h}\right)^2, \quad \sigma_{\theta,u}^* = \frac{\sigma_{\theta,u}}{E}\left(\frac{R}{h}\right)^2, \tag{6.159}$$

we obtain

$$\sigma_{r,u}^* = \frac{2a\zeta}{1-\nu^2}\left[(1+\nu)-b(3+\nu)\frac{r^2}{R^2}\right], \\ \sigma_{\theta,u}^* = \frac{2a\zeta}{1-\nu^2}\left[(1+\nu)-b(1+3\nu)\frac{r^2}{R^2}\right]. \tag{6.160}$$

From (6.156) we can determine the rigidity δ_3 ((6.125)) of the stiffening rib at which no compressive stresses σ_θ appear along $r = R$ ($\sigma_\theta^* \geqslant 0$), so that buckling cannot occur. This will be the case for $\delta_3 \geqslant \frac{1}{\nu}$. Values of $q^*(\zeta)$, obtained from (6.149) for a plate reinforced by an inextensible rib ($\delta_3 = \infty$, $\delta_1 \geqslant 0$), are given in Table 29 for different rigidities δ_1, with $\nu = 0{,}3$.

TABLE 29. Values of $q^*(\zeta)$

ζ	δ_1							
	0	0.5	1	2	5	10	25	∞
0.5	1.050	1.229	1.381	1.627	2.088	2.465	2.859	3.275
1	4.095	4.383	4.639	5.074	5.954	6.741	7.620	8.622
1.5	11.12	11.38	11.64	12.16	13.37	14.64	16.18	18.11
2	24.14	24.16	24.28	24.70	26.13	27.96	30.44	33.82
2.5	45.12	44.62	44.41	44.51	45.99	48.52	52.30	57.81
3	76.08	74.71	73.92	73.42	74.74	78.14	83.65	89.15
3.5	119.0	116.3	114.7	113.2	114.2	118.6	126.4	138.9
4	175.9	171.4	168.6	165.8	166.0	171.7	182.5	200.2
4.5	248.7	241.9	237.4	232.9	232.1	239.4	253.7	278.1
5	339.4	329.7	323.2	316.4	314.2	323.3	342.1	374.6

It is seen that an inextensible stiffening rib (points on the contour $r = R$ are fixed) markedly influences the rigidity of the plate only at small deflections. Furthermore, when $\xi \geqslant 2.5$, we find that at many values of δ_1 the rigidity of the reinforced plate is less than that of the nonreinforced

plate. This seems unlikely. Hence, approximation by the first term only of the expansion (6.131) of the deflection w is insufficiently accurate when a stiffening rib is provided.

Case of concentrated load

The plate is loaded by a concentrated force P_1 at the center and reinforced by an elastic rib along the support contour $r = R$. We solve this problem by Bubnov and Galerkin's method, by representing the function w ((6.131)) as the deflections (2.284) of a thick plate with reinforced edge under the same load:

$$w = f\left(1 - \frac{r^2}{R^2} + 2b\frac{r^2}{R^2}\ln\frac{r}{R}\right) = f\eta_1, \tag{6.161}$$

where

$$b = \frac{1+\nu+\delta_1}{3+\nu+\delta_1}. \tag{6.162}$$

Substituting (6.161) in (6.137), we find

$$\begin{aligned}\frac{d\Phi}{dr} = -\frac{Ef^2r^3}{R^4(3+\nu+\delta_1)^2}\Big[&(1+\nu+\delta_1)^2\left(\ln^2\frac{r}{R} - \frac{3}{2}\ln\frac{r}{R} + \frac{7}{8}\right) + 1 - \\ &- 2(1+\nu+\delta_1)\left(\ln\frac{r}{R} - \frac{3}{4}\right)\Big] + \\ &+ \frac{Ef^2r}{R^2(3+\nu+\delta_1)^2[1+(1-\nu)\delta_3]}\Big\{1 + \\ &+ \frac{7}{8}(1+\nu+\delta_1)^2 + \frac{3}{2}(1+\nu+\delta_1) + \Big[\frac{9-7\nu}{8}(1+\nu+\delta_1)^2 + \\ &+ \frac{5-3\nu}{2}(1+\nu+\delta_1) + (3-\nu)\Big]\delta_3\Big\},\end{aligned} \tag{6.163}$$

while (6.111) becomes

$$\psi = P/2\pi r. \tag{6.164}$$

Substituting (6.163) and (6.164) first in (6.133), and then in (6.132), we obtain:

$$k_1\zeta^3 + k_2\zeta = P^*, \tag{6.165}$$

where

$$k_2 = \frac{4a}{3(1-\nu^2)(3+\nu+\delta_1)}, \quad a = (1+\nu+\delta_1), \quad P^* = \frac{PR^2}{\pi Eh^4}, \tag{6.166}$$

$$k_1 = -\frac{8}{(3+\nu+\delta_1)^4}\left[\frac{47}{648}a^4+\frac{47}{108}a^3+\frac{141}{108}a^2+\frac{13}{9}a+\frac{2}{3}\right]+$$

$$+\frac{8\left(1+\frac{a}{2}+\frac{a^2}{8}\right)}{(3+\nu+\delta_1)^4[1+(1-\nu)\delta_3]}\left\{\left[\frac{9-7\nu}{8}a^2+\frac{5-3\nu}{2}a+3-\nu\right]\delta_3+\right. \quad (6.166)\ (\text{cont'd})$$

$$\left.+\frac{7}{8}a^2+\frac{3}{2}a+1\right\}.$$

Assigning different limiting values to the parameters δ_1 and δ_3, we obtain the values of k_1 and k_2 in (6.165) for various support conditions of the plate (cf. the preceding example).

The stresses in the middle surface are at $r = 0$:

$$\sigma_r^* = \sigma_\theta^* = \frac{\zeta^2}{(3+\nu+\delta_1)^2[1+(1-\nu)\delta_3]}\left\{\frac{7}{8}a^2+\frac{3}{2}a+1+\right.$$

$$\left.+\left[\frac{9-7\nu}{8}a^2+\frac{5-3\nu}{2}a+(3-\nu)\right]\delta_3\right\}. \quad (6.187)$$

At $r = R$

$$\sigma_r^* = \frac{\zeta^2(a^2+4a+8)\delta_3}{4(3+\nu+\delta_1)^2[1+(1-\nu)\delta_3]},$$

$$\sigma_\theta^* = \frac{\zeta^2(a^2+4a+8)(\nu\delta_3-1)}{4(3+\nu+\delta_1)^2[1+(1-\nu)\delta_3]}. \quad (6.168)$$

Refined solution by the perturbation method for a uniformly loaded circular plate

A uniformly loaded circular plate with elastic support rib has already been considered, using f as single parameter in the expression (6.145) for the deflections w. A different procedure will be employed in this section, based on the perturbation method.

Let the magnitudes w^*, σ_r^*, q^* ((6.138)) be given in the form (6.141). We require that the first approximation w_1^*, which is a solution of (6.142), satisfy the boundary conditions (6.114) at $r = R$. Substituting (6.132) in (6.114), we obtain

$$\left.\begin{aligned}(1+\delta_1+\nu)\frac{dw_1^*}{dx}-2\frac{d^2w_1^*}{dx^2}&=0\\ w_1^*&=0\end{aligned}\right\}\ \text{at}\ x=0. \quad (6.169),\ (6.170)$$

In addition, the derivative $\frac{dw_1^*}{dx}$ must be finite at the center of the plate:

$$\left[(1-x)\frac{dw_1^*}{dx}\right]_{x=1}=0. \quad (6.171)$$

Since at the center $w^* = \zeta$, we obtain

$$(w_1^*)_{x=1} = 1. \quad (6.172)$$

Integrating (6.142) twice, we obtain

$$(1-x)\frac{dw_1^*}{dx} = -2a_1x^2 + C_1x + C_2. \tag{6.173}$$

From (6.171) and (6.169) we find

$$C_1 = \frac{2(\delta_1 - 1 + \nu)\alpha_1}{1+\nu+\delta_1}, \quad C_2 = \frac{4a_1}{1+\nu+\delta_1}. \tag{6.174}$$

Substitution of (6.174) in (6.173) yields

$$\frac{dw_1^*}{dx} = 2a_1\left(x + \frac{2}{1+\nu+\delta_1}\right). \tag{6.175}$$

Integrating (6.175) and using (6.170) and (6.172), we obtain

$$w_1^* = \frac{(1+\nu+\delta_1)x^2}{5+\nu+\delta_1} + \frac{4x}{5+\nu+\delta_1}, \quad a_1 = \frac{1+\nu+\delta_1}{5+\nu+\delta_1}. \tag{6.176}$$

The deflections w_1^* obtained coincide with the deflections (2.281) of a thick plate with reinforced edge.

Substituting for w_1^* from (6.176) in the right-hand side of (6.143), we obtain as second approximation

$$\frac{d^2}{dx^2}[(1-x)g_2(x)] = -\frac{1}{2}\left(\frac{dw_1^*}{dx}\right)^2 = -2\frac{a^2}{b^2}x^2 - 8\frac{a}{b^2}x - \frac{8}{b^2}, \tag{6.177}$$

where

$$a = 1+\nu+\delta_1, \quad b = 5+\nu+\delta_1. \tag{6.178}$$

Integration of (6.177) yields

$$(1-x)g_2(x) = -\frac{2}{b^2}\left(\frac{1}{12}a^2x^4 + \frac{2}{3}ax^3 + 2x^2\right) + A'x + B'. \tag{6.179}$$

The constants A' and B' are determined from (6.124) and from the condition (6.126) that the stresses σ_r at the plate center be finite. Substituting (6.138), these conditions become:

$$\left.\begin{aligned} \left(1-\nu+\frac{1}{\delta_2}\right)\sigma_r^* - 2(1-x)\frac{d\sigma_r^*}{dx} = 0 \quad &\text{at } x=0 \\ \sqrt{1-x}\ \ \sigma_r^* = 0 \quad &\text{at } x=1 \end{aligned}\right\}. \tag{6.180}$$

We require that $g_2(x)$ satisfy (6.180):

$$\left.\begin{aligned} \left(1-\nu+\frac{1}{\delta_2}\right)g_2(x) - 2(1-x)\frac{dg_2(x)}{dx} = 0 \quad &\text{at } x=0 \\ \sqrt{1-x}\,g_2(x) = 0 \quad &\text{at } x=1 \end{aligned}\right\}. \tag{6.181}$$

We obtain finally

$$A' = \frac{(a^2 + 8a + 24)[1 - (1 + \nu)\delta_3]}{6b^2[1 + (1 - \nu)\delta_3]}, \quad B' = \frac{(a^2 + 8a + 24)\delta_3}{3b^2[1 + (1 - \nu)\delta_3]}. \tag{6.182}$$

Hence,

$$g_2(x) = \frac{1}{6b^2}\left[\frac{2(a^2 + 8a + 24)\delta_3}{1 + (1 - \nu)\delta_3} + a^2(1 + x + x^2)x + 8a(1 + x)x + 24x\right]. \tag{6.183}$$

Substituting for w_1^* from (6.176) and for $g_2(x)$ from (6.183) in the right-hand side of (6.144), we obtain the third approximation for w_3^*, which must satisfy the boundary conditions (6.169) and (6.170) at the support rib:

$$w_3^* = 0, (1 + \nu + \delta_1)\frac{dw_3^*}{dx} - 2\frac{d^2w_3^*}{dx^2} = 0 \text{ at } x = 0. \tag{6.184}$$

In addition, we must have at the center of the plate (cf. (6.171)):

$$(1 - x)\frac{dw_3^*}{dx} = 0 \text{ for } x = 1. \tag{6.185}$$

Taking (6.172) into account, we also obtain

$$w_3^* = 0 \text{ for } x = 1. \tag{6.186}$$

Integrating (6.144) and determining the integration constants and α_3 from the boundary conditions (6.184) to (6.186), we obtain finally

$$\begin{aligned}
\alpha_3 &= \frac{1 - \nu^2}{360b^4}\left[73a^4 + 776a^3 + 3300a^2 + 6720a + 5760 + \right. \\
&\left. + \frac{20\delta_3}{1 + (1 - \nu)\delta_3}(5a^4 + 70a^3 + 432a^2 + 1296a + 1728)\right], \\
w_3^* &= \frac{1 - \nu^2}{360b^4}[(43a^3 + 356a^2 + 1020a + 960)(ax^2 + 4x) - \\
&- 2ba^3x^5 - 6ba^2(a + 6)x^5 - 15b(a^3 + 8a^2 + 20a)x^4 - 20b(a^3 + 10a^2 + \\
&+ 36a + 48)x^3] + \frac{(1 - \nu^2)ax\delta_3}{9b^4[1 + (1 - \nu)\delta_3]}(a^2 + 8a + 24)(ax + 4 - bx^2).
\end{aligned} \tag{6.187}$$

Proceeding only to the third approximation, we find from (6.141):

$$\alpha_1\zeta + \alpha_3\zeta^3 = \frac{3}{16}q^*(1 - \nu^2), \tag{6.188}$$

where α_1 and α_3 are given by (6.176) and (6.187).

The stresses σ_r^* and $\sigma_\theta^* = \frac{d}{dr}(r\sigma_r^*)$ in the middle plane are [by (6.183)]:

$$\sigma_r^* = \frac{\zeta^2}{6b^2}\left[a^2(x^3 + x^2 + x) + 8a(x^2 + x) + 24x + \frac{2}{1 - \nu + \delta_3^{-1}}(a^2 + 8a + 24)\right],$$

$$\sigma_\theta^* = \frac{1}{6b^3}\Big[a^2(7x^3 - x^2 - x - 2) + 8a(5x^2 - x - 2) + 24(3x - 2) +$$

$$+ \frac{2}{1 - \nu + \delta_3^{-1}}(a^2 + 8a + 24)\Big]\zeta^2.$$

The bending stresses are given by (cf. (6.159)):

$$\sigma_{r,u}^* = \frac{2}{b(1-\nu)}\left[ax + 2 - \frac{2a(1-x)}{1+\nu}\right]\zeta - \left\{\frac{1+\nu}{30b^3}a^3x^5 +\right.$$

$$+ \frac{1+\nu}{12b^3}(a^3 + 6a^2)x^4 + \frac{1+\nu}{6b^3}(a^3 + 8a^2 + 20a)x^3 +$$

$$+ \frac{1+\nu}{6b^3}\left[a^3 + 10a^2 + 36a + 48 + \frac{2}{1 - \nu + \delta_3^{-1}}(a^3 + 8a^2 + 24a)\right]x^2 -$$

$$- \frac{1+\nu}{180b^4}\left[43a^4 + 356a^3 + 1020a^2 + 960a + \frac{40}{1 - \nu + \delta_3^{-1}}(a^4 + 8a^3 +\right.$$

$$\left. + 24a^2)\right]x - \frac{1+\nu}{90b^4}\left[43a^3 + 356a^2 + 1020a + 960 +\right.$$

$$\left. + \frac{40}{1 - \nu + \delta_3^{-1}}(a^3 + 8a^2 + 24a)\right] - (1-x)x^4\frac{a^3}{3b^3} -$$

$$- \frac{2}{3b^3}(a^3 + 6a^2)(1-x)x^3 - \frac{1}{b^3}(a^3 + 8a^2 + 20a)(1-x)x^2 -$$

$$- \frac{2}{3b^3}\left[a^3 + 10a^2 + 36a + 48 + \frac{2}{1 - \nu + \delta_3^{-1}}(a^3 + 8a^2 + 24a)\right](1-x)x +$$

$$\left. + \frac{1-x}{90b^4}\left[43a^4 + 356a^3 + 1020a^2 + 960a + \frac{40}{1 - \nu + \delta_3^{-1}}(a^4 + 8a^3 + 24a^2)\right]\right\}\zeta^3,$$

$$\sigma_{\theta,u}^* = \frac{2}{b(1-\nu)}\left[ax + 2 - \frac{2a\nu}{1+\nu}(1-x)\right]\zeta - \left\{\frac{1+\nu}{30b^3}a^3x^5 +\right.$$

$$+ \frac{1+\nu}{12b^3}(a^3 + 6a^2)x^4 + \frac{1+\nu}{6b^3}(a^3 + 8a^2 + 20a)x^3 + \frac{1+\nu}{6b^3}\left[a^3 + 10a^2 +\right.$$

$$\left. + 36a + 48 + \frac{2}{1 - \nu + \delta_3^{-1}}(a^3 + 8a^2 + 24a)\right]x^2 - \frac{1+\nu}{180b^4}\left[43a^4 +\right.$$

$$+ 356a^3 + 1020a^2 + 960a + \frac{40}{1 - \nu + \delta_3^{-1}}(a^4 + 8a^3 + 24a^2)\Big]x -$$

$$- \frac{1+\nu}{90b^4}\left[43a^3 + 356a^2 + 1020a + 960 + \frac{40}{1 - \nu + \delta_3^{-1}}(a^3 + 8a^2 +\right.$$

$$\left. + 24a)\right] - \frac{\nu a^3}{3b^3}(1-x)x^4 - \frac{2\nu}{3b^3}(a^3 + 6a^2)(1-x)x^4 -$$

$$- \frac{\nu}{b^3}(a^3 + 8a^2 + 24a)(1-x)x^3 - \frac{2\nu}{3b^3}\left[a^3 + 10a^2 + 36a + 48 +\right.$$

$$\left. + \frac{2}{1 - \nu + \delta_3^{-1}}(a^3 + 8a^2 + 24a)\right]x(1-x) + \frac{\nu(1-x)}{90b^4}\left[43a^4 +\right.$$

$$\left. + 356a^3 + 1020a^2 + 960a + \frac{40}{1 - \nu + \delta_3^{-1}}(a^4 + 8a^3 + 24a^2)\right]\Big\}\zeta^3.$$

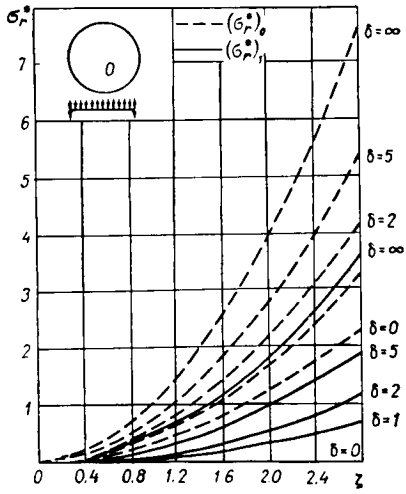


FIGURE 95.

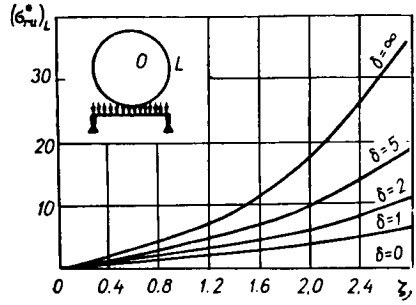


FIGURE 96.

The influence of the stiffening rib on the stresses in the plate can be appraised from the values of σ_r^*, σ_θ^*, $\sigma_{r,u}^*$, $\sigma_{\theta,u}^*$, calculated for a rib of rectangular cross section (width b_1, height h_1), in which case

$$\delta_1 = \frac{E_1}{E}\cdot\frac{b_1}{R}\left(\frac{h_1}{h}\right)^3(1-\nu^2),\quad \delta_3 = \frac{E_1}{12E}\cdot\frac{h_1}{h}\left(\frac{b_1}{R}\right)^3\left(1+12\frac{R^2}{b_1^2}\right).$$

The following assumptions were made: $\nu = 0.3$, $\frac{h_1}{h} = 3$, $\frac{b_1}{R} = 0.1$. Different values between 0 and ∞ were assigned to $\delta = \frac{E_1}{E}$, which determines the relative rigidity of the rib. Figures 95 to 98* give the stresses at the center and at the edge of the plate as functions of ζ for different values of δ. The curves of the stresses $\sigma_{r,u}^*$ in Figure 96 were plotted for the upper surface of the plate, and the curves of the stresses $\sigma_{\theta,u}^*$ in Figure 97 for the lower surface.

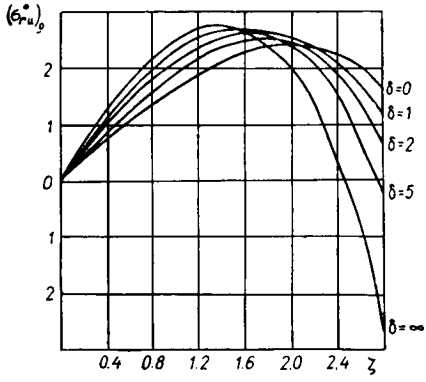


FIGURE 97.

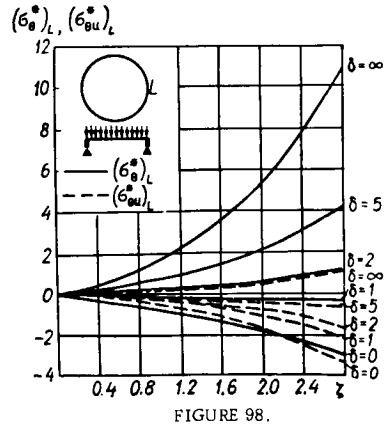


FIGURE 98.

* The calculations were performed by P. L. Katz in his diploma thesis.

It is seen from Figures 95, 96, and 98 that for fixed values of ζ, $(\sigma_r^*)_{x=0}$ and $(\sigma_{r,u}^*)_{x=0}$ increase with the rigidity of the support rib, while the absolute values of $(\sigma_\theta^*)_{x=0}$ and $(\sigma_{\theta,u}^*)_{x=0}$ first become zero, and then increase. When $\delta > 4$, $\sigma_{r,u}^*$ first vanishes at the center of the plate for $\zeta > 2.45$ (Figure 97), and then becomes negative. This contradicts the experimental results /17/. Higher-order approximations are thus necessary to calculate the bending stresses.

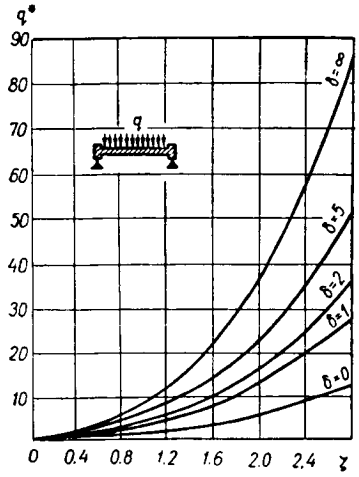


FIGURE 99.

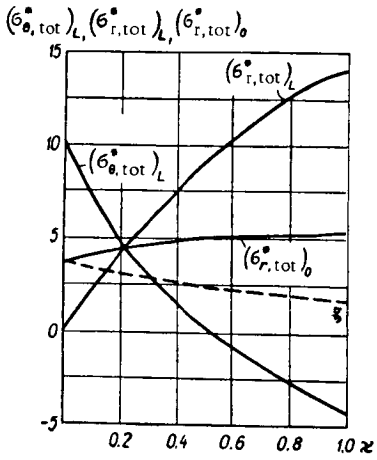


FIGURE 100.

We determined the influence of the stiffening rib on the deflections of the plate by plotting $q^*(\zeta)$ for different values of $\delta = \frac{E_1}{E}$ (Figure 99). If the rib consists of the same material as the plate ($\delta = 1$), a load $q^* = 13.09$ causes the same dimensionless deflection $\zeta = 2.8$ at the center as a load $q^* = 27.68$ applied to a nonreinforced plate.

Figure 100 gives, for $q^* = 25$, the maximum total normal stresses $\sigma_{r,\text{tot}}^*$ and $\sigma_{\theta,\text{tot}}^*$ at two points 0 and L of the plate (cf. Figure 96) as functions of $\varkappa = \frac{\delta}{\delta + 2}$. The influence of the support-rib rigidity on the stresses in the plate at a fixed load is clearly seen. The broken line in Figure 100 represents the dimensionless deflection ζ at the plate center as function of $\varkappa$ for $q^* = 25$.

From the stress curves (Figure 100) we can select the optimum value of δ with respect to strength. In our case the stresses become minimum at $\varkappa \approx 0.22$, i. e., $\delta \approx 0.56$; $(\sigma_{r,\text{tot}}^*)_L \approx (\sigma_{r,\text{tot}}^*)_0 \approx 4.45$. This is 56% less than the theoretical stress $(\sigma_{\theta,\text{tot}}^*)_L = 10.27$ in the nonreinforced plate. When $\varkappa \approx 0.22$, the maximum deflection at the center is 17% less than that of a plate without rib.

We can solve similarly the problem of reinforcing the inner edge of an annular plate, using the boundary conditions (6.115) and (6.124).

§ 7. SOME OTHER PROBLEMS CONCERNING PLATES WITH CURVED STIFFENING RIBS

Many authors have considered the influence of curved* elastic ribs on the stresses and strains of plates.

In most studies undertaken before 1950 the elastic rib reinforcing the edge of a hole in a thin plate was considered to be wide, the stresses in it being given by Sophie German's equation.

This formulation applied mainly to problems concerning simple or built-up ribs with uniform cross sections around circular holes. Problems concerning reinforcements around noncircular holes usually give rise to considerable mathematical difficulties which impose severe restrictions on the shapes of the contours to be chosen.

The main results obtained by the various authors are given in /8/, /118/, and /187/.

The above-mentioned mathematical model cannot be used when the elastic stiffening element is thin or has an irregular profile. The mathematical model for the stiffening rib is in such cases usually a curved thin elastic bar of uniform cross section (cf. § 1 of Chapter IV).

The problem of bending of an unbounded plate having a hole with reinforced curved edge can be solved in this formulation by Sheremet'ev's method of successive approximations /187/; solutions were thus obtained for elliptical as well as for triangular and square holes with rounded corners (/71/ to /73/). Generalizing Sheremet'ev's method, Martynovich /74/ obtained an approximate solution of the problem of bending of a thin plate occupying a doubly-connected region forming a confocal elliptic ring with reinforced edges.

Unbounded anisotropic plates reinforced around circular holes were considered in /187/ and /190/.

Bending of circular and annular plates with concentric stiffening ribs was discussed in /28, 31, 53, 101, 105, 106, 131/, etc. In /131/ the stiffening ribs, arranged nonsymmetrically with respect to the middle plane of the plate, were in the axisymmetrical case considered as short cylindrical shells of uniform thickness. Passing to the limit, solutions were obtained for cases in which the stiffening element can be considered as an unbounded shell or thin ring. The range of application of the various solutions was also considered.

The saving in weight achieved by reinforcing the plate was studied in /27/ for a case of axisymmetrical loading of an annular plate reinforced by annular or radial ribs.

The influence of annular ribs on the carrying capacities of symmetrically loaded circular and annular plates was considered in detail in /14, 15/, /174, 175, 175a/.

Determination of the critical forces in the case of symmetrical buckling of circular and annular plates with concentric thin ribs was discussed in /13/.

* We disregard here the numerous papers dealing with the influence of straight ribs on the strength, rigidity, stability, and vibrations of thick and thin plates (cf. for instance, /17/, /18/, /19/, /59/, /136/, /137/, etc.).

Chapter VII

SHELLS WITH STIFFENING RIBS

§ 1. BOUNDARY CONDITIONS FOR A SHELL WITH A HOLE WHOSE EDGE IS REINFORCED BY A THIN ELASTIC RIB

Consider a thin isotropic shell* with a hole whose edge is reinforced by a thin elastic stiffening rib whose axial line Γ lies in the middle surface of the shell.

Let $\vec{n}, \vec{b}, \vec{\tau}$ he a system of orthogonal axes at each point on Γ. The $\vec{n}$ axis is normal to the section of the shell along Γ, the $\vec{b}$ axis is normal to the middle surface of the shell, and the $\vec{\tau}$ axis is tangent to Γ. The axes $\vec{\xi}, \vec{\eta}, \vec{\zeta}$ of the principal trihedron of Γ form with the $\vec{n}, \vec{b}, \vec{\tau}$ axes angles (Figure 101) whose cosines are given in Table 30.

TABLE 30

	$\vec{n}$	$\vec{b}$	$\vec{\tau}$
$\vec{\xi}$	m_1	l_1	0
$\vec{\eta}$	$-l_1$	m_1	0
$\vec{\zeta}$	0	0	-1

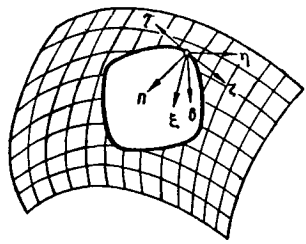


FIGURE 101.

The theory of small deformations of thin bars possessing double curvature / 66, 87, 102/, leads to the following equations describing the deformation of the stiffening rib:

1) The first group of Clebsch's equations, by which the extensibility of the axial line of the rib is approximately taken into account:

$$\theta_\eta = \frac{d}{ds} u_\xi^0 + \omega_\eta u_\zeta^0 - \omega_\zeta u_\eta^0, \tag{7.1}$$

$$-\theta_\xi = \frac{d}{ds} u_\eta^0 + \omega_\zeta u_\xi^0 - \omega_\xi u_\zeta^0, \tag{7.2}$$

$$\varepsilon_0 = \frac{d}{ds} u_\zeta^0 + \omega_\xi u_\eta^0 - \omega_\eta u_\xi^0, \tag{7.3}$$

* An extensive list of papers on the theory of shells, published in the USSR up to 1957, can be found in Oniashvili's survey /91/. For surveys devoted to particular problems of the theory of shells, see Bolotin /4/, Kil'chevskii /51/, Kovalenko /358/, Mushtari /84/, Savin /119-121/, and others.

where $\vec{\theta} = \theta_\xi \vec{i} + \theta_\eta \vec{j} + \theta_\zeta \vec{k}$ is the vector of rotation of the moving trihedron ξ, η, ζ; u_ξ^0, u_η^0, u_ζ^0 are the projections on the corresponding axes of the displacement vectors of points on Γ; ε_0 is the unit elongation of Γ; s is the arc length measured on Γ.

2) The second group of Clebsch's equations

$$\left.\begin{aligned} \delta\omega_\xi = \frac{d}{ds}\theta_\xi + \omega_\eta\theta_\zeta - \omega_\zeta\theta_\eta, \quad \delta\omega_\eta = \frac{d}{ds}\theta_\eta + \omega_\zeta\theta_\xi - \omega_\xi\theta_\zeta \\ \delta\omega_\zeta = \frac{d}{ds}\theta_\zeta + \omega_\xi\theta_\eta - \omega_\eta\theta_\xi \end{aligned}\right\}, \tag{7.4}$$

where ω_ξ, ω_η, ω_ζ are the principal components of the curvature and torsion of the rib, and $\delta\omega_\xi$, $\delta\omega_\eta$, $\delta\omega_\zeta$ are their corresponding increments.

3) The equilibrium conditions

$$\left.\begin{aligned} \frac{d}{ds}V_\xi + \omega_\eta V_\zeta - \omega_\zeta V_\eta + p_\xi = 0 \\ \frac{d}{ds}V_\eta + \omega_\xi V_\xi - \omega_\xi V_\zeta + p_\eta = 0 \\ \frac{d}{ds}V_\zeta + \omega_\xi V_\eta - \omega_\eta V_\xi + p_\zeta = 0 \end{aligned}\right\}, \tag{7.5}$$

$$\left.\begin{aligned} \frac{d}{ds}L_\xi + \omega_\eta L_\zeta - \omega_\zeta L_\eta - V_\eta = 0 \\ \frac{d}{ds}L_\eta + \omega_\zeta L_\xi - \omega_\xi L_\zeta + V_\xi = 0 \\ \frac{d}{ds}L_\zeta + \omega_\xi L_\eta - \omega_\eta L_\xi + m_\zeta = 0 \end{aligned}\right\}, \tag{7.6}$$

where $\vec{V}(V_\xi, V_\eta, V_\zeta)$ and $\vec{L}(L_\xi, L_\eta, L_\zeta)$ are the principal force vector and the principal moment acting in section s of the rib; $\vec{p}(p_\xi, p_\eta, p_\zeta)$ and $\vec{m}$ $(0, 0, m_\zeta)$ are the applied force and bending moment acting on the rib, per unit length of its axial line.

4) Kirchhoff's equation

$$\delta\omega_\xi = \frac{L_\xi}{A}, \quad \delta\omega_\eta = \frac{L_\eta}{B}, \quad \delta\omega_\zeta = \frac{L_\zeta}{C}, \tag{7.7}$$

where A and B are the flexural rigidities of the rib, referred to the $\vec{\xi}$ and $\vec{\eta}$ axes; C is the torsional rigidity of the rib.

In accordance with Hooke's law we assume approximately

$$\varepsilon_0 = \frac{V_\zeta}{E_1 F}, \tag{7.8}$$

where E_1F is the tensile rigidity of the rib.

The boundary conditions at the joint Γ between shell and rib are:

$$u_\xi = u_\xi^0, \quad u_\eta = u_\eta^0, \quad u_\zeta = u_\zeta^0, \tag{7.9}$$

$$\theta_\zeta = \frac{\partial w}{\partial n} + \frac{u_n}{R_2'} - \frac{u_\zeta}{R_{12}}, \tag{7.10}$$

$$S' = -p_n + S^0, \qquad T' = p_\zeta + T^0,$$

$$Q_n^* = -p_b + Q^0, \qquad M_n = m_\zeta + M^0, \tag{7.11}$$

where S' and T' are, respectively, the reduced normal and tangential forces, Q_n^* is the generalized tranverse (shearing) force, and M_n the generalized bending moment acting in the shell section whose normal is $\vec{n}$; p_n, p_ζ, p_b, m_ζ are the corresponding forces and moments acting on the rib; $\frac{1}{R_2'}$ is the curvature of the normal section of the shell through the normal to Γ; $\frac{1}{R_{12}}$ is (/26/, p. 100) given by (7.58) below; h is the thickness of the shell; the displacement vector of points on the middle surface of the shell is

$$\vec{u}(u_n, w, u_\tau) = \vec{u}(u_\xi, u_\eta, u_\zeta). \tag{7.12}$$

From (7.9) and Table 30 we find

$$u_\xi^0 = m_1 u_n + l_1 w, \quad u_\eta^0 = m_1 w - l_1 u_n, \quad u_\zeta^0 = -u_\tau. \tag{7.13}$$

Substituting (7.13) in (7.1) to (7.3) and (7.10), we obtain

$$\left.\begin{aligned} \theta_\eta &= -\omega_\eta u_\tau - \omega_\zeta (m_1 w - l_1 u_n) + \frac{\partial}{\partial s}(m_1 u_n + l_1 w) \\ \theta_\xi &= -\omega_\zeta (m_1 u_n + l_1 w) - \omega_\zeta u_\tau - \frac{\partial}{\partial s}(m_1 w - l_1 u_n) \\ \theta_\zeta &= \frac{\partial w}{\partial n} + \frac{u_n}{R_2'} + \frac{u_\tau}{R_{12}} \end{aligned}\right\}, \tag{7.14}$$

$$\varepsilon_0 = \omega_\xi (m_1 w - l_1 u_n) - \omega_\eta (m_1 u_n + l_1 w) - \frac{\partial}{\partial s} u_\tau. \tag{7.15}$$

Substitution of (7.7) in (7.4) yields

$$\left.\begin{aligned} L_\xi &= A\left(\frac{d}{ds}\theta_\xi + \omega_\eta \theta_\zeta - \omega_\zeta \theta_\eta\right) \\ L_\eta &= B\left(\frac{d}{ds}\theta_\eta + \omega_\zeta \theta_\xi - \omega_\xi \theta_\zeta\right) \\ L_\zeta &= C\left(\frac{d}{ds}\theta_\zeta + \omega_\xi \theta_\eta - \omega_\eta \theta_\xi\right) \end{aligned}\right\}. \tag{7.16}$$

Equating the right-hand sides of (7.8) and (7.15), we obtain

$$V_\zeta = E_1 F\left(\omega_n w - \omega_b u_n - \frac{\partial}{\partial s} u_\tau\right), \tag{7.17}$$

where

$$\omega_n = m_1\omega_\xi - l_1\omega_\eta, \quad \omega_b = l_1\omega_\xi + m_1\omega_\eta.$$

The projections on the $\vec{n}$ and $\vec{b}$ axes of the forces p_ξ and p_η are

$$p_n = m_1 p_\xi - l_1 p_\eta, \quad p_b = l_1 p_\xi + m_1 p_\eta. \tag{7.18}$$

Substituting in (7.5) for V_ξ, V_η, and V_ζ from the first two equations (7.6) and from (7.17), we obtain p_ξ, p_η, and p_ζ expressed through L_ξ, L_η, L_ζ, and V_ζ.

In particular, the third equation (7.5) yields

$$p_\zeta = \left(\omega_\zeta\omega_\xi - \omega_\eta\frac{\partial}{\partial s}\right)L_\eta - \left(\omega_\zeta\omega_\eta + \omega_\xi\frac{\partial}{\partial s}\right)L_\xi - \frac{\partial}{\partial s}V_\zeta. \tag{7.19}$$

We obtain similar expressions for p_ξ and p_η, whose substitution in (7.18) yields

$$\left.\begin{aligned} p_n = &-\omega_b V_\zeta - \Phi_2\left[\omega_\zeta L_\xi - \omega_\xi L_\zeta + \frac{\partial}{\partial s}L_\eta\right] - \\ &- \Phi_1\left[\omega_\zeta L_\eta - \omega_\eta L_\zeta - \frac{\partial}{\partial s}L_\xi\right] \\ p_b = &\ \omega_n V_\zeta + \Phi_1\left[\omega_\zeta L_\xi - \omega_\xi L_\zeta + \frac{\partial}{\partial s}L_\eta\right] - \\ &- \Phi_2\left[\omega_\zeta L_\eta - \omega_\eta L_\zeta - \frac{\partial}{\partial s}L_\xi\right] \end{aligned}\right\}, \tag{7.20}$$

where

$$\Phi_1[\ \ ] = \left(\omega_\zeta m_1 + l_1\frac{\partial}{\partial s}\right)[\ \ ], \quad \Phi_2[\ \ ] = \left(\omega_\zeta l_1 - m_1\frac{\partial}{\partial s}\right)[\ \ ]. \tag{7.21}$$

In addition, the last equation (7.6) yields:

$$m_\zeta = \omega_\eta L_\xi - \omega_\xi L_\eta - \frac{\partial}{\partial s}L_\zeta. \tag{7.22}$$

Substituting (7.19) to (7.22) in (7.11), we obtain the following boundary conditions on Γ (/157, 160/):

$$\left.\begin{aligned} S' &= S^0 + \omega_b V_\zeta + \Phi_2\left[\omega_\zeta L_\xi - \omega_\xi L_\zeta + \frac{\partial}{\partial s}L_\eta\right] + \Phi_1\left[\omega_\zeta L_\eta - \omega_\eta L_\zeta - \frac{\partial}{\partial s}L_\xi\right] \\ T' &= T^0 - \frac{\partial}{\partial s}V_\zeta + \left(\omega_\zeta\omega_\xi - \omega_\eta\frac{\partial}{\partial s}\right)L_\eta - \left(\omega_\zeta\omega_\eta + \omega_\xi\frac{\partial}{\partial s}\right)L_\xi \\ M_n &= M^0 + \omega_\eta L_\xi - \omega_\xi L_\eta - \frac{\partial}{\partial s}L_\zeta \\ Q_n^* &= Q^0 - \omega_n V_\zeta - \Phi_1\left[\omega_\zeta L_\xi - \omega_\xi L_\zeta + \frac{\partial}{\partial s}L_\eta\right] + \Phi_2\left[\omega_\zeta L_\eta - \omega_\eta L_\zeta - \frac{\partial}{\partial s}L_\xi\right] \end{aligned}\right\}. \tag{7.23}$$

The right-hand sides of (7.23) contain the magnitudes L_ξ, L_η, L_ζ, V_ζ, which by (7.16) and (7.17) are expressed through the displacement components u_n, u_τ, w. The left-hand sides of (7.13) are determined from the displacement components by the formulas of the theory of shells (cf., for instance, /26/ or /16/).

Equations (7.23) thus represent the boundary conditions for the three components of the displacement vector at the joint between the shell and a thin elastic rib.

Boundary conditions are similarly derived for the case of a thin elastic rib reinforcing the outer edge of a shell. If the stiffening rib is located anywhere on the middle surface (not at the edge of the shell), the boundary conditions for the parts of the shell adjoining the rib will have the same form (7.23). In this case, S', T', Q_n^* and M_n^* are the jumps across Γ, in the corresponding forces and bending moments per unit length. In addition, we must then add to the four conditions (7.23):

$$w^+ = w^-,\quad u_n^+ = u_n^-,\quad u_\tau^+ = u_\tau^-,\quad \theta_\xi^+ = \theta_\xi^- \text{ on } \Gamma. \tag{7.24}$$

§ 2. CASE OF SHALLOW SHELL

In the case of a shallow shell we can make the simplifying assumption that the binormal $\vec{b}$ to Γ remains fixed in direction (this assumption is exact for a plane curve). Hence, the torsion of the axial line of the rib is zero.

We also assume that m_1 and l_1 (cf. Table 30) are constant, so that the torsion of the rib [as a whole] is also zero /66/. The projections of the vector $\vec{\omega}$ on the ξ, η, ζ axes are then

$$\omega_\xi = \frac{l_1}{\varrho},\quad \omega_\eta = \frac{m_1}{\varrho},\quad \omega_\zeta = 0, \tag{7.25}$$

where $\frac{1}{\varrho}$ is the curvature of the rib.

Substituting (7.14), (7.16), (7.17), and (7.25) in (7.23), we obtain* the boundary conditions on the joint Γ between the shallow shell and the thin rib:

$$S = S^0 - \frac{E_1F}{\varrho}\left(\frac{\partial u_\tau}{\partial s} + \frac{u_n}{\varrho}\right) + \frac{\partial}{\partial s}\left[\beta'\frac{\partial^2}{\partial s^2}\left(\frac{u_\tau}{\varrho} - \frac{\partial u_n}{\partial s}\right) + \gamma\frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial s^2} - \frac{1}{\varrho}\cdot\frac{\partial w}{\partial n}\right)\right], \tag{7.26}$$

$$T = T_0 + E_1F\frac{\partial}{\partial s}\left(\frac{\partial u_\tau}{\partial s} + \frac{u_n}{\varrho}\right) + \frac{1}{\varrho}\left[\beta'\frac{\partial^2}{\partial s^2}\left(\frac{u_\tau}{\varrho} - \frac{\partial u_n}{\partial s}\right) + \gamma\frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial s^2} - \frac{1}{\varrho}\cdot\frac{\partial w}{\partial n}\right)\right], \tag{7.27}$$

* In the expression (7.10) for θ_ξ we have neglected the small terms $\frac{u_n}{R_2'}$ and $\frac{u_\tau}{R_{12}}$, as is usually done in the theory of thin shallow shells (/16/, Chapter VII).

$$M_n = M^0 - C\frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial n \partial s} + \frac{1}{\varrho}\frac{\partial w}{\partial s}\right) + \frac{1}{\varrho}\left[\beta\left(\frac{1}{\varrho}\cdot\frac{\partial w}{\partial n} - \frac{\partial^2 w}{\partial s^2}\right) - \right.$$
$$\left. - \gamma\frac{\partial}{\partial s}\left(\frac{u_\tau}{\varrho} - \frac{\partial u_n}{\partial s}\right)\right], \tag{7.28}$$

$$Q_n^* = Q^0 - \beta\frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial s^2} - \frac{1}{\varrho}\frac{\partial w}{\partial n}\right) + \frac{\partial}{\partial s}\left[C\frac{1}{\varrho}\left(\frac{\partial^2 w}{\partial n \partial s} + \frac{1}{\varrho}\cdot\frac{\partial w}{\partial s}\right) - \right.$$
$$\left. - \gamma\frac{\partial^2}{\partial s^2}\left(\frac{u_\tau}{\varrho} - \frac{\partial u_n}{\partial s}\right)\right], \tag{7.29}$$

$$\beta' = m_1^2 B + l_1^2 A, \quad \beta = m_1^2 A + l_1^2 B, \quad \gamma = l_1 m_1 (A - B). \tag{7.30}$$

Here S and T are, respectively, the normal and tangential forces in the shallow shell at Γ. The boundary conditions on Γ have the form (7.26) to (7.29) also in the case of a thin plate, whose edge is reinforced by a thin elastic rib, if neither principal axis of inertia of a rib cross section lies in the middle plane of the plate.

We shall consider some particular cases.

1) The rib is perfectly flexible, i.e., the edge of the shallow shell is not reinforced. Setting in (7.26) to (7.29) $A = B = C = E_1F = 0$, i.e., $\beta' = \beta = \gamma = E_1F = 0$, we obtain

$$S = S^0, \; T = T^0, \; M_n = M^0, \; Q_n^* = Q^0, \tag{7.31}$$

as ought to be expected.

2) The rib is perfectly rigid, i.e., $A = B = C = E_1F = \infty$. The necessary conditions for S, T, M_n and $\int_0^s Q_n^* ds$ to be finite are in this case:

$$\frac{\partial u_\tau}{\partial s} + \frac{u_n}{\varrho} = 0, \quad \frac{\partial}{\partial s}\left(\frac{u_\tau}{\varrho} - \frac{\partial u_n}{\partial s}\right) = 0, \tag{7.32}$$

$$\frac{\partial^2 w}{\partial n \partial s} + \frac{1}{\varrho}\cdot\frac{\partial w}{\partial s} = 0, \quad \frac{\partial^2 w}{\partial s^2} - \frac{1}{\varrho}\cdot\frac{\partial w}{\partial n} = 0. \tag{7.33}$$

Integrating the second equation (7.32) with respect to s, we find

$$\frac{u_\tau}{\varrho} - \frac{\partial u_n}{\partial s} = \varepsilon, \tag{7.34}$$

where ε is a constant. From (7.34) and the first equation (7.32) we form the complex expression

$$\frac{\partial}{\partial s}(u_n + iu_\tau) + \frac{i}{\varrho}(u_n + iu_\tau) = -\varepsilon. \tag{7.35}$$

We have

$$(u_n + iu_\tau)\dot{t} = i(u + iv),$$

where u and v are the projections of $\vec{u}$ ((7.12)) on the x- and y-axes fixed in the plan of the shallow shell. Using (2.16) in its complex conjugate form we can rewrite (7.35) as follows:

$$\frac{\partial}{\partial s}(u + iv) = i\varepsilon\dot{t}.$$

Integration with respect to s yields

$$u + iv = i\varepsilon(x + iy) + (u_0 + iv_0) \text{ on } \Gamma, \tag{7.36}$$

where $(u_0 + iv_0)$ is a complex constant.

We similarly rewrite (7. 33), first as

$$\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial n} + i\frac{\partial w}{\partial s}\right) - \frac{i}{\varrho}\left(\frac{\partial w}{\partial n} + i\frac{\partial w}{\partial s}\right) = 0,$$

and then as

$$\frac{\partial}{\partial s}\left[\dot{i}\left(\frac{\partial w}{\partial n} + i\frac{\partial w}{\partial s}\right)\right] = 0 \text{ on } \Gamma.$$

Integration with respect to s yields

$$\frac{\partial w}{\partial n} + i\frac{\partial w}{\partial s} = (C_1 + iC_2)(\dot{x} + i\dot{y}) \text{ on } \Gamma,$$

where $(C_1 + iC_2)$ is a constant. Hence

$$\theta_i = \frac{\partial w}{\partial n} = C_1\dot{x} - C_2\dot{y}, \tag{7.37}$$

$$\frac{\partial w}{\partial s} = C_1\dot{y} + C_2\dot{x}. \tag{7.38}$$

Integration of (7.38) yields

$$w = C_1 y + C_2 x + C_3 \text{ on } \Gamma, \tag{7.39}$$

where C_3 is a constant.

Equations (7.36), (7.37), and (7.39) are the boundary conditions for a shell whose edge is reinforced by a perfectly rigid rib along Γ. As ought to be expected, these conditions define only rigid-body displacements and rotations. The coefficients ε, u_0, v_0, C_1, C_2, C_3 are determined from the six equilibrium conditions for the rib.

3) Consider now the case when one of the principal axes of inertia of each rib cross section lies in the middle surface of the shallow shell. Then

$$l_1 = 0, \quad m_1 = 1, \quad \gamma = 0, \quad \beta = A, \quad \beta' = B \tag{7.40}$$

and the boundary conditions (7.26) to (7.29) reduce to

$$\left.\begin{aligned}
S &= S^0 + B\frac{\partial^3}{\partial s^3}\left(\frac{u_\tau}{\varrho} - \frac{\partial u_n}{\partial s}\right) - \frac{E_1F}{\varrho}\left(\frac{\partial u_\tau}{\partial s} + \frac{u_n}{\varrho}\right)\\
T &= T^0 + B\frac{1}{\varrho}\cdot\frac{\partial^2}{\partial s^2}\left(\frac{u_\tau}{\varrho} - \frac{\partial u_n}{\partial s}\right) + E_1F\frac{\partial}{\partial s}\left(\frac{\partial u_\tau}{\partial s} + \frac{u_n}{\varrho}\right)\\
M_n &= M^0 - C\frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial n\,\partial s} + \frac{1}{\varrho}\cdot\frac{\partial w}{\partial s}\right) + \frac{A}{\varrho}\left(\frac{1}{\varrho}\cdot\frac{\partial w}{\partial n} - \frac{\partial^2 w}{\partial s^2}\right)\\
Q_n^* &= Q^0 + \frac{\partial}{\partial s}\left[C\cdot\frac{1}{\varrho}\left(\frac{\partial^2 w}{\partial n\,\partial s} + \frac{1}{\varrho}\cdot\frac{\partial w}{\partial s}\right) + A\frac{\partial}{\partial s}\left(\frac{1}{\varrho}\cdot\frac{\partial w}{\partial n} - \frac{\partial^2 w}{\partial s^2}\right)\right]
\end{aligned}\right\}. \tag{7.41}$$

The last two conditions are, of course, identical with (2.355).

4) If, in addition to (7.40), we assume that Γ forms a circle of radius $\varrho = \varrho_0 = \text{const.}$, we obtain from (7.41)

$$\left.\begin{aligned} S_\varrho &= S^0 - B\frac{1}{\varrho_0^4}\cdot\frac{\partial^3}{\partial\lambda^3}\left(u_\lambda - \frac{\partial u_\varrho}{\partial\lambda}\right) + E_1F\cdot\frac{1}{\varrho_0^2}\left(\frac{\partial u_\lambda}{\partial\lambda} + u_\varrho\right)\\ S_{\varrho\lambda} &= T^0 - B\frac{1}{\varrho_0^4}\cdot\frac{\partial^2}{\partial\lambda^2}\left(u_\lambda - \frac{\partial u_\varrho}{\partial\lambda}\right) - E_1F\cdot\frac{1}{\varrho_0^2}\frac{\partial}{\partial\lambda}\left(\frac{\partial u_\lambda}{\partial\lambda} + u_\varrho\right)\\ M_\varrho &= M^0 + C\frac{1}{\varrho_0^2}\cdot\frac{\partial}{\partial\lambda}\left(\frac{\partial^2 w}{\partial\varrho\partial\lambda} - \frac{1}{\varrho_0}\cdot\frac{\partial w}{\partial\lambda}\right) -\\ &\quad - A\frac{1}{\varrho_0^2}\left(\frac{\partial w}{\partial\varrho} + \frac{1}{\varrho_0}\cdot\frac{\partial^2 w}{\partial\lambda^2}\right)\\ Q_\varrho^* &= Q^0 + \frac{1}{\varrho_0^3}\cdot\frac{\partial}{\partial\lambda}\left[C\left(\frac{\partial^2 w}{\partial\varrho\partial\lambda} - \frac{1}{\varrho_0}\cdot\frac{\partial w}{\partial\lambda}\right) +\right.\\ &\quad \left. + A\frac{\partial}{\partial\lambda}\left(\frac{\partial w}{\partial\varrho} + \frac{1}{\varrho_0}\cdot\frac{\partial^2 w}{\partial\lambda^2}\right)\right] \end{aligned}\right\}, \qquad (7.42)$$

where the forces and moments are represented in polar coordinates (ϱ, λ), with origin at the center of the hole $(\vec{r} = -\vec{n},\ \vec{\lambda} = -\vec{\tau})$.

5) If the stiffening rib forms a support, (7.29) must be replaced by the condition $w = 0$, and (7.26) to (7.28) are simplified by setting $\frac{\partial w}{\partial s} = \frac{\partial^2 w}{\partial s^2} = 0$. In this case (7.29) serves for determining the reactions along the support contour.

§ 3. TRANSFORMATION OF BOUNDARY CONDITIONS

The boundary conditions (7.26) to (7.29) for a shallow shell can also be represented in terms of the stress function $\Phi(x, y)$ and the deflections $w(x, y)$ /16/. Using (2.16) we write,

$$\frac{\partial}{\partial s}(u_n + iu_\tau) = \frac{\partial}{\partial s}[i\dot{\bar{t}}\,(u + iv)] = i\dot{\bar{t}}\,\frac{\partial}{\partial s}(u + iv) - \frac{i}{\varrho}[i\dot{\bar{t}}\,(u + iv)], \qquad (7.43)$$

where u, v are the projections of the displacement vector on the x- and y-axes, respectively; ϱ is the radius of curvature of Γ.

Hence,

$$\left(\frac{\partial u_n}{\partial s} - \frac{u_\tau}{\varrho}\right) + i\left(\frac{\partial u_\tau}{\partial s} + \frac{u_n}{\varrho}\right) = i\dot{\bar{t}}\,\frac{\partial}{\partial s}(u + iv). \qquad (7.44)$$

Remembering that

$$\frac{\partial}{\partial s} = \dot{x}\,\frac{\partial}{\partial x} + \dot{y}\,\frac{\partial}{\partial y},$$

we separate the real and imaginary parts in (7.44):

$$\left.\begin{aligned}\frac{\partial u_n}{\partial s}-\frac{u_\tau}{\varrho}&=\dot{x}\dot{y}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\dot{y}^2\frac{\partial u}{\partial y}-\dot{x}^2\frac{\partial v}{\partial x}\\ \frac{\partial u_\tau}{\partial s}+\frac{u_n}{\varrho}&=\dot{x}\dot{y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\dot{y}^2\frac{\partial v}{\partial y}+\dot{x}^2\frac{\partial u}{\partial x}\end{aligned}\right\}. \tag{7.45}$$

For a shallow shell we have

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\frac{1}{E}\left(\frac{\partial^2\Phi}{\partial y^2}-\nu\frac{\partial^2\Phi}{\partial x^2}\right)+\frac{w}{\varrho_x}\\ \frac{\partial v}{\partial y}&=\frac{1}{E}\left(\frac{\partial^2\Phi}{\partial x^2}-\nu\frac{\partial^2\Phi}{\partial y^2}\right)+\frac{w}{\varrho_y}\\ \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}&=-\frac{2(1+\nu)}{E}\cdot\frac{\partial^2\Phi}{\partial x\partial y}\end{aligned}\right\}, \tag{7.46}$$

where $\frac{1}{\varrho_x}$ and $\frac{1}{\varrho_y}$ are the curvatures of the middle surface in the x - and y-directions, respectively. Substituting (7.46) in (7.45), we find

$$\left.\begin{aligned}\frac{\partial u_\tau}{\partial s}+\frac{u_n}{\varrho}&=\frac{1}{E}L_1[\Phi]+w\left(\frac{\dot{x}^2}{\varrho_x}+\frac{\dot{y}^2}{\varrho_y}\right)\\ \frac{\partial}{\partial s}\left(\frac{\partial u_n}{\partial s}-\frac{u_\tau}{\varrho}\right)&=\frac{1}{\varrho E}L_2[\Phi]+\dot{x}\frac{\partial}{\partial y}\left(\frac{w}{\varrho_x}\right)-\dot{y}\frac{\partial}{\partial x}\left(\frac{w}{\varrho_y}\right)+\\ &+\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}\right)\left[\dot{x}\dot{y}\left(\frac{w}{\varrho_x}-\frac{w}{\varrho_y}\right)\right]\end{aligned}\right\}, \tag{7.47}$$

where

$$\left.\begin{aligned}L_1[\Phi]&=(\dot{y}^2-\nu\dot{x}^2)\frac{\partial^2\Phi}{\partial x^2}+(\dot{x}^2-\nu\dot{y}^2)\frac{\partial^2\Phi}{\partial y^2}-2(1+\nu)\,\dot{x}\dot{y}\frac{\partial^2\Phi}{\partial x\partial y}\\ L_2[\Phi]&=\varrho\dot{x}(1+\dot{y}^2+\nu\dot{y}^2)\frac{\partial^3\Phi}{\partial y^3}-\varrho\dot{y}(1+\dot{x}^2+\nu\dot{x}^2)\frac{\partial^3\Phi}{\partial x^3}-\\ &-\varrho\dot{x}[1+2\nu-3(1+\nu)\dot{x}^2]\frac{\partial^3\Phi}{\partial x^2\partial y}+\\ &+\varrho\dot{y}[1+2\nu-3(1+\nu)\dot{y}^2]\frac{\partial^3\Phi}{\partial x\partial y^2}+\\ &+(1+\nu)\left[(\dot{x}^2-\dot{y}^2)\left(\frac{\partial^2\Phi}{\partial y^2}-\frac{\partial^2\Phi}{\partial x^2}\right)-4\,\dot{x}\dot{y}\frac{\partial^2\Phi}{\partial x\partial y}\right]\end{aligned}\right\}. \tag{7.48}$$

It follows that (7.26) to (7.29) can be written as follows:

$$\left.\begin{aligned}S&=S^0-\frac{E_1F}{\varrho}\left\{\frac{1}{E}L_1[\Phi]+L_3[w]\right\}+\\ &+\frac{\partial}{\partial s}\left\{\gamma\frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial s^2}-\frac{1}{\varrho}\cdot\frac{\partial w}{\partial n}\right)-\beta'\frac{\partial}{\partial s}\left(\frac{1}{\varrho E}L_2[\Phi]+L_4[w]\right)\right\}\end{aligned}\right| \tag{7.49}$$

$$
\left.
\begin{aligned}
T = T^0 + E_1 F \frac{\partial}{\partial s}\left\{\frac{1}{E} L_1[\Phi] + L_3[w]\right\} + \\
+ \frac{1}{\varrho}\left\{\gamma \frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial s^2} - \frac{1}{\varrho}\cdot\frac{\partial w}{\partial n}\right) - \beta' \frac{\partial}{\partial s}\left(\frac{1}{\varrho E} L_2[\Phi] + L_4[w]\right)\right\} \\
M_n = M^0 + \frac{\gamma}{\varrho}\left\{\frac{1}{\varrho E} L_2[\Phi] + L_4[w]\right\} + \\
+ \frac{\beta}{\varrho}\left(\frac{1}{\varrho}\cdot\frac{\partial w}{\partial n} - \frac{\partial^2 w}{\partial s^2}\right) - C\frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial n \partial s} + \frac{1}{\varrho}\cdot\frac{\partial w}{\partial s}\right) \\
Q_n^* = Q^0 + \gamma \frac{\partial^2}{\partial s^2}\left\{\frac{1}{\varrho E} L_2[\Phi] + L_4[w]\right\} + \\
+ \frac{\partial}{\partial s}\left[\frac{C}{\varrho}\left(\frac{\partial^2 w}{\partial n \partial s} + \frac{1}{\varrho}\cdot\frac{\partial w}{\partial s}\right)\right] - \beta \frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial s^2} - \frac{1}{\varrho}\cdot\frac{\partial w}{\partial n}\right)
\end{aligned}
\right\}, \qquad (7.49)\ \text{cont'd}
$$

where

$$
\left.
\begin{aligned}
L_4[w] = \dot{x}\frac{\partial}{\partial y}\left(\frac{w}{\varrho_x}\right) - \dot{y}\frac{\partial}{\partial x}\left(\frac{w}{\varrho_y}\right) + \\
+ \left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}\right)\left[\dot{x}\dot{y}\left(\frac{w}{\varrho_x} - \frac{w}{\varrho_y}\right)\right] \\
L_3[w] = w\left(\frac{\dot{x}^2}{\varrho_x} + \frac{\dot{y}^2}{\varrho_y}\right)
\end{aligned}
\right\}. \qquad (7.50)
$$

For a spherical shallow shell with $\varrho_x = \varrho_y = R$ we have $L_4[w] = \frac{1}{R}\cdot\frac{\partial w}{\partial n}$, $L_3[w] = \frac{w}{R}$. The terms containing w in the first two equations (7.46) should be omitted in the case of a thin plate for which the boundary conditions are obtained from (7.49) by setting $L_4[w] = L_3[w] = 0$.

§ 4. DIFFERENT WAY OF DERIVING THE BOUNDARY CONDITIONS FOR A SHELL REINFORCED AT THE EDGE BY A THIN RIB

Let the shell edge form the smooth closed curve Γ which corresponds to the curve $\beta = \text{const.}$ in the plane system of curvilinear coordinates (α, β). A thin elastic rib is welded to the shell along this edge. The shell is loaded arbitrarily.

By the principle of virtual displacements we have for the system shell-rib

$$
\delta V_0 + \delta V_R - \delta W = 0, \qquad (7.51)
$$

where δV_0 is the variation of the elastic energy of the shell; δV_R is the variation of the elastic energy of the thin rib, considered as a thin bar of double curvature, δW is the work done by the load on virtual displacements of the shell.

Let us visualize the rib severed from the shell, and consider the equilibrium of the latter when subjected to the specified load and to the

unknown forces and moments of interaction with the rib, applied along Γ. By the principle of virtual displacements

$$\delta V_0 - \delta W - \delta W_\Gamma = 0, \tag{7.52}$$

where δW_Γ is the work done by the elastic forces and moments with which the rib acts on the shell along Γ, on the corresponding virtual displacements and rotations of the middle surface.

Subtracting (7.52) from (7.51), we obtain

$$\delta V_R + \delta W_\Gamma = 0. \tag{7.53}$$

We write (/26/, Chapters II and IV)

$$\delta W_\Gamma = \int_\Gamma (\vec{R}\delta\vec{U} + \vec{Q}\delta\vec{\Omega})\, ds, \tag{7.54}$$

where $\vec{R}$ is the principal force vector and $\vec{Q}$ is the principal moment acting in the normal section of the shell along Γ; $\delta\vec{U}$ is the vector of the virtual elastic translations of points on the middle surface; $\delta\Omega$ is the vector of the virtual elastic rotations:

$$\begin{gathered}\vec{R} = S_n\vec{n} - T_n\vec{\zeta} + Q_n\vec{b}, \quad \vec{Q} = H\vec{n} - M_n\vec{\zeta}, \\ \delta\vec{U} = \delta u_\zeta\cdot\vec{\zeta} + \delta u_n\vec{n} + \delta w\cdot\vec{b}, \quad \delta\vec{\Omega} = \delta\theta_\zeta\cdot\vec{\zeta} + \delta\theta_n\cdot\vec{n} + \delta\theta_b\vec{b},\end{gathered} \tag{7.55}$$

where $\vec{\zeta}$, $\vec{n}$, $\vec{b}$ are three orthogonal unit vectors; the direction of Γ is along the tangent to Γ, that of $\vec{n}$ along the normal to the section of the shell.

Substituting (7.55) in (7.54), we obtain

$$\delta W_\Gamma = \int_\Gamma (S_n\delta u_n - T_n\delta u_\zeta + Q_n\delta w - M_n\delta\theta_\zeta + H\delta\theta_n)\, ds. \tag{7.56}$$

We have

$$\delta\theta_n = -\frac{\partial}{\partial s}(\delta w) - \frac{\delta u_\zeta}{R_1'} + \frac{\delta u_n}{R_{12}}, \tag{7.57}$$

where $\frac{1}{R_1'}$ is the curvature of the normal section along Γ of the middle surface of the shell,

$$\frac{1}{R_{12}} = \frac{\sin 2\chi}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \tag{7.58}$$

R_1, R_2 are the principal radii of curvature; χ is the angle between Γ and the direction in which the normal curvature of the surface is $\frac{1}{R_1}$.

Substitution of (7.57) in (7.56) yields

$$\delta W_\Gamma = \int_\Gamma \left[\left(S_n + \frac{H}{R_{12}} \right) \delta u_n - \left(T_n + \frac{H}{R_1'} \right) \delta u_\zeta + Q_n \delta w - M_n \delta\theta_\zeta \right] ds - \int_\Gamma H \frac{\partial}{\partial s} (\delta w)\, ds. \tag{7.59}$$

Integrating the last term by parts, assuming δw and H to be continuous on the closed contour Γ, we obtain

$$\delta W_\Gamma = \int_\Gamma (S_n' \delta u_n - T_n' \delta u_\zeta + Q_n^* \delta w - M_n \delta\theta_\zeta) ds, \tag{7.60}$$

where

$$S_n' = S_n + \frac{H}{R_{12}}, \quad T_n' = T_n + \frac{H}{R_1'}, \quad Q_n^* = Q_n + \frac{\partial H}{\partial s}. \tag{7.61}$$

Neglecting the terms depending on the transverse forces, the elastic energy of a thin rib is (/103/, Vol. I, p. 580):

$$V_R = \frac{1}{2} \int_\Gamma \left(\frac{V_\xi^2}{E_1 F} + \frac{L_\xi^2}{A} + \frac{L_\eta^2}{B} + \frac{L_\zeta^2}{C} \right) ds, \tag{7.62}$$

whence

$$\delta V_R = \int_\Gamma \left(\frac{V_\zeta}{E_1 F} \delta V_\zeta + \frac{L_\xi}{A} \delta L_\xi + \frac{L_\eta}{B} \delta L_\eta + \frac{L_\zeta}{C} \delta L_\zeta \right) ds. \tag{7.63}$$

Substituting for L_ξ, L_η, L_ζ from (7.16) and for V_ζ from (7.8) and (7.3), using (7.9), we obtain

$$\begin{aligned} \delta V_R = \int_\Gamma & V_\zeta (\omega_n \delta w - \omega_b \delta u_n) + L_\xi (\omega_\eta \delta\theta_\zeta - \omega_\zeta \delta\theta_\eta) + \\ & + L_\eta (\omega_\zeta \delta\theta_\xi - \omega_\xi \delta\theta_\zeta) + L_\zeta (\omega_\xi \delta\theta_\eta - \omega_\eta \delta\vartheta_\xi)] ds + I_1, \end{aligned} \tag{7.64}$$

where

$$I_1 = \int_\Gamma \left[V_\zeta \frac{d}{ds} (\delta u_\zeta) + L_\xi \frac{d}{ds} (\delta\theta_\xi) + L_\eta \frac{d}{ds} (\delta\theta_\eta) + L_\zeta \frac{d}{ds} (\delta\theta_\zeta) \right] ds, \tag{7.65}$$

$$\left. \begin{aligned} \omega_n &= m_1 \omega_\xi - l_1 \omega_\eta, \quad \omega_b = l_1 \omega_\xi + m_1 \omega_\eta \\ \delta u_\xi &= m_1 \delta u_n + l_1 \delta w, \quad \delta u_\eta = m_1 \delta w - l_1 \delta u_n \end{aligned} \right\}, \tag{7.66}$$

$$l_1 = \cos(\xi, b), \quad m_1 = \cos(\xi, n). \tag{7.67}$$

It is assumed that the functions in the integrand of (7.65) are continuous on the closed contour Γ. Integrating by parts, we find

$$I_1 = -\int_\Gamma \left(\frac{dV_\zeta}{ds} \delta u_\zeta + \frac{dL_\xi}{ds} \delta\theta_\xi + \frac{dL_\eta}{ds} \delta\theta_\eta + \frac{dL_\zeta}{ds} \delta\theta_\zeta \right) ds. \tag{7.68}$$

We can thus rewrite (7.64) as follows:

$$\delta V_R = \int_l \left[V_\zeta (\omega_n \delta\omega - \omega_b \delta u_n) - \frac{\partial V_\zeta}{\partial s} \delta u_\zeta + (\omega_\eta L_\xi - \omega_\xi L_\eta - \frac{\partial L_\zeta}{\partial s}) \delta\theta_\zeta + \right.$$
$$\left. + \left(\omega_\zeta L_\eta - \omega_\eta L_\zeta - \frac{\partial L_\xi}{\partial s}\right) \delta\theta_\xi + \left(\omega_\xi L_\zeta - \omega_\zeta L_\xi - \frac{\partial L_\eta}{\partial s}\right) \delta\theta_\eta \right] ds. \tag{7.69}$$

The magnitudes $\delta\theta_\xi$ and $\delta\theta_\eta$ are found by varying Clebsch's equations (7.1) and (7.2). Substituting the result in (7.69), we obtain

$$\delta V_R = \int_l \left[V_\zeta (\omega_n \delta\omega - \omega_b \delta u_n) - \frac{\partial V_\zeta}{\partial s} \delta u_\zeta + \left(\omega_\eta L_\xi - \omega_\xi L_\eta - \frac{\partial L_\zeta}{\partial s}\right) \delta\theta_\zeta + \right.$$
$$+ \left(\omega_\zeta L_\eta - \omega_\eta L_\zeta - \frac{\partial L_\xi}{\partial s}\right)(\omega_\xi \delta u_\zeta - \omega_\zeta \delta u_\xi) + \left(\omega_\xi L_\zeta - \right.$$
$$\left. - \omega_\zeta L_\xi - \frac{\partial L_\eta}{\partial s}\right)(\omega_\eta \delta u_\zeta - \omega_\zeta \delta u_\eta) \Big] ds + \tag{7.70}$$
$$+ \int_l \left[\left(\omega_\eta L_\zeta - \omega_\zeta L_\eta + \frac{\partial L_\xi}{\partial s}\right) \frac{d}{ds} (\delta u_\eta) + \right.$$
$$\left. + \left(\omega_\xi L_\zeta - \omega_\zeta L_\xi - \frac{\partial L_\eta}{\partial s}\right) \frac{d}{ds} (\delta u_\xi) \right] ds.$$

Integration by parts of the second integral in (7.70) yields

$$\delta V_R = \int_l \left[\left(\omega_\zeta \omega_\xi - \omega_\eta \frac{\partial}{\partial s}\right) L_\eta - \left(\omega_\zeta \omega_\eta + \omega_\xi \frac{\partial}{\partial s}\right) L_\xi - \frac{\partial V_\zeta}{\partial s} \right] \delta u_\zeta ds -$$
$$- \int_l \left\{ \omega_b V_\zeta + \Phi_2 \left[\omega_\zeta L_\xi - \omega_\xi L_\zeta + \frac{\partial L_\eta}{\partial s} \right] + \right.$$
$$\left. + \Phi_1 \left[\omega_\zeta L_\eta - \omega_\eta L_\zeta - \frac{\partial L_\xi}{\partial s} \right] \right\} \delta u_\zeta ds + \tag{7.71}$$
$$+ \int_l \left(\omega_\eta L_\xi - \omega_\xi L_\eta - \frac{\partial L_\zeta}{\partial s}\right) \delta\theta_\zeta ds + \int_l \left\{ \omega_n V_\zeta + \right.$$
$$\left. + \Phi_1 \left[\omega_\zeta L_\xi - \omega_\xi L_\zeta + \frac{\partial L_\eta}{\partial s} \right] - \Phi_2 \left[\omega_\zeta L_\eta - \omega_\eta L_\zeta - \frac{\partial L_\xi}{\partial s} \right] \right\} \delta\omega ds.$$

where $\Phi_1[\ ]$ and $\Phi_2[\ ]$ are given by (7.21).

We substitute (7.71) and (7.60) in (7.53), and equate to zero the expressions in braces before the independent virtual displacements and rotations δu_ζ, $\delta\theta_\zeta$, δu_n, and $\delta\omega$. We thus obtain the following four boundary conditions at the joint between the shell and the thin rib:

$$\left.\begin{aligned} T_n' &= \left(\omega_\zeta \omega_\xi - \omega_\eta \frac{\partial}{\partial s}\right) L_\eta - \left(\omega_\zeta \omega_\eta + \omega_\xi \frac{\partial}{\partial s}\right) L_\xi - \frac{\partial}{\partial s} V_\zeta \\ S_n' &= \omega_b V_\zeta + \Phi_2 \left[\omega_\zeta L_\xi - \omega_\xi L_\zeta + \frac{\partial}{\partial s} L_\eta \right] + \\ &+ \Phi_1 \left[\omega_\zeta L_\eta - \omega_\eta L_\zeta - \frac{\partial}{\partial s} L_\xi \right] \end{aligned}\right\} \tag{7.72}$$

$$\left.\begin{aligned} M_n &= \omega_\eta L_\xi - \omega_\xi L_\eta - \frac{\partial}{\partial s} L_\zeta \\ Q_n^* &= -\omega_n V_\zeta - \Phi_1\left[\omega_\zeta L_\xi - \omega_\xi L_\zeta + \frac{\partial}{\partial s} L_\eta\right] + \\ &\quad + \Phi_2\left[\omega_\zeta L_\eta - \omega_\eta L_\zeta - \frac{\partial L_\xi}{\partial s}\right] \end{aligned}\right\}. \qquad \begin{matrix}(7.72)\\(\text{cont'd})\end{matrix}$$

If external forces T^0, S^0, Q^0 and moments M^0 are applied to the rib, we must in (7.72) replace

$$\left.\begin{aligned} T_n' \text{ by } T_n' - T^0, \quad S_n' \text{ by } S_n' - S^0 \\ Q_n^* \text{ by } Q_n^* - Q^0, \quad M_n \text{ by } M_n - M^0 \end{aligned}\right\}. \qquad (7.73)$$

Conditions (7.72) then become identical with (7.23).

§ 5. INFLUENCE OF A STIFFENING RIB ON THE STRESS CONCENTRATIONS IN A CYLINDRICAL SHELL WITH CIRCULAR HOLE

As example of the use of the boundary conditions derived in § 2 for a shell with reinforced edge, we shall consider the stress distribution near a "circular" hole (l') of small radius $\varrho = \varrho_0$ located on the mantle of a cylindrical shell of thickness h (Figure 102). The edge of the hole is reinforced by a thin elastic rib. One of the principal axes of inertia of each rib cross section lies in the middle surface of the shell. The shell is loaded by a uniform internal pressure p_0. This problem was solved by A.I. Lur'e /67/ for a nonreinforced shell; our solution is based on his work (/154, 156/). When the shell has no hole, the stresses are $S_1 = ph$, $S_2 = qh$, where $2p = q = \frac{p_0 R}{h}$ (R is the radius of the cylinder).

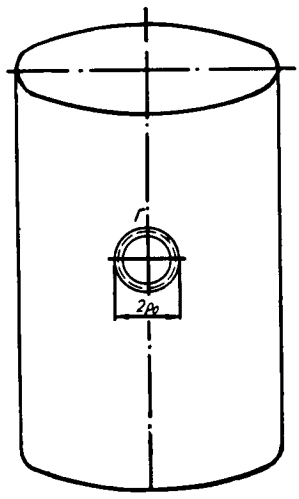


FIGURE 102.

We solve the problem (/ 68/, Chapter IV, §§ 8, 9) by determining the complex stress function σ which characterizes the additional state of stress due to the hole and rib. Using polar coordinates $(\varrho\lambda)$, the moments M_ϱ, M_λ and the forces S_ϱ, $S_{\varrho\lambda}$, S_λ, Q_ϱ^* in the shell are

$$\left.\begin{aligned} S_\varrho &= \frac{1}{2}h(p+q)+\frac{1}{2}h(p-q)\cos 2\lambda - \\ &- \frac{D}{h}\sqrt{12(1-\nu^2)}\left(\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\lambda^2}\right)\operatorname{Im}\sigma \\ S_{\varrho\lambda} &= \frac{1}{2}h(q-p)\sin 2\lambda+\frac{D}{h}\sqrt{12(1-\nu^2)}\left[\frac{\partial}{\partial\varrho}\left(\frac{1}{\varrho}\cdot\frac{\partial}{\partial\lambda}\right)\right]\operatorname{Im}\sigma \\ S_\lambda &= \frac{1}{2}h(p+q)-\frac{1}{2}h(p-q)\cos 2\lambda-\frac{D}{h}\sqrt{12(1-\nu^2)}\frac{\partial^2}{\partial\varrho^2}\operatorname{Im}\sigma \end{aligned}\right\}. \tag{7.74}$$

$$\left.\begin{aligned} M_\varrho &= -D\left(\frac{\partial^2}{\partial\varrho^2}+\frac{\nu}{\varrho}\cdot\frac{\partial}{\partial\varrho}+\frac{\nu}{\varrho^2}\cdot\frac{\partial^2}{\partial\lambda^2}\right)\operatorname{Re}\sigma \\ M_\lambda &= -D\left(\nu\frac{\partial^2}{\partial\varrho^2}+\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\lambda^2}\right)\operatorname{Re}\sigma \\ Q_\varrho^* &= -D\left[\frac{\partial}{\partial\varrho}\Delta+(1-\nu)\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho}\left(\frac{1}{\varrho}\cdot\frac{\partial^2}{\partial\lambda^2}\right)\right]\operatorname{Re}\sigma \end{aligned}\right\}, \tag{7.75}$$

where

$$\Delta=\frac{\partial^2}{\partial\varrho^2}+\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\lambda^2},\quad D=\frac{Eh^3}{12(1-\nu^2)},$$

E is Young's modulus, ν is Poisson's ratio.

With an accuracy up to squares of the small parameter $\beta=\sqrt[4]{3(1-\nu^2)}\times\frac{1}{2\sqrt{Rh}}$ we can write

$$\begin{aligned} \operatorname{Im}\sigma &= \frac{2}{\pi}A_0\left(\ln\frac{\varrho}{\varrho_0}+\gamma'\right)-\frac{1}{\pi}C_0(1+\cos 2\lambda)-\frac{2}{\pi}F_1\cdot\frac{1}{\varrho^2}\cos 2\lambda+ \\ &+\beta^2\left[\frac{1}{4}C_0\varrho^2(1+\cos 2\lambda)-\frac{\varrho^2}{4}A_0(2+\cos 2\lambda)+\right. \\ &\left.+\frac{2}{\pi}A_1\left(\ln\frac{\varrho}{\varrho_0}+\gamma'\right)+\frac{1}{2}B_1-\frac{2}{\pi}F_2\cdot\frac{1}{\varrho^2}\cos 2\lambda-\frac{1}{\pi}C_1(1+\cos 2\lambda)\right], \end{aligned} \tag{7.76}$$

$$\begin{aligned} w=\operatorname{Re}\sigma=\frac{\beta^2}{\pi}\Bigg\{&\frac{\pi}{2}A_1-2\gamma'B_1+D_1-\frac{1}{2}F_1+A_0\left(2\varrho^2\ln\frac{\varrho}{\varrho_0}+\right. \\ &\left.+2\gamma'\varrho^2-\varrho^2\right)+2A_2\varrho^2\ln\frac{\varrho}{\varrho_0}-2B_1\ln\frac{\varrho}{\varrho_0}+C_0\varrho^2\left(\frac{1}{4}-\ln\frac{\varrho}{\varrho_0}-\gamma'\right)+ \\ &+\left[A_0\varrho^2\left(\ln\frac{\varrho}{\varrho_0}+\gamma'\right)+C_0\varrho^2\left(\frac{1}{6}-\ln\frac{\varrho}{\varrho_0}-\gamma'\right)+D_1-\frac{2}{\varrho^2}E_2-\right. \\ &\left.-\frac{4}{\varrho^2}H_2\right]\cos 2\lambda-\left(\frac{\varrho^2}{12}C_0+\frac{1}{2}F_1+\frac{4}{\varrho^2}H_2+\frac{24}{\varrho^4}M_3\right)\cos 4\lambda\Bigg\}, \end{aligned} \tag{7.77}$$

where

$$\gamma' = \ln \frac{\gamma \varrho_0 \beta}{\sqrt{2}}, \quad \ln \gamma = 0.577216. \tag{7.78}$$

The function σ differs from Lur'e's function /68/ only by the real term $2A_2 \frac{\beta^2}{\pi} \varrho^2 \ln \frac{\varrho}{\varrho_0}$. The 12 unknown coefficients of the function σ will be determined with the aid of (7.42). Let the cover over the hole be of such design that only shearing forces are transmitted to the reinforced edge of the shell. Hence, in (7.42), we can set

$$S^0 = T^0 = M^0 = 0, \quad Q^0 = -\frac{p_0 \varrho_0}{2}, \tag{7.79}$$

the shearing forces Q^0 being in equilibrium with the resultant of the pressure p_0 acting on the area of the hole.

We shall first express the displacement components u_ϱ and u_λ through the stress function σ. Setting

$$x = \varrho \cos \lambda, \quad y = \varrho \sin \lambda, \tag{7.80}$$

we transform the strain components in cartesian coordinates (/68/, p. 182)

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y} + \frac{w}{R}, \quad \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \tag{7.81}$$

into the strain components in polar coordinates

$$\varepsilon_\varrho = \frac{\partial u_\varrho}{\partial \varrho} + \frac{w}{R} \sin^2 \lambda, \tag{7.82}$$

$$\varepsilon_\lambda = \frac{1}{\varrho} \cdot \frac{\partial u_\lambda}{\partial \lambda} + \frac{u_\varrho}{\varrho} + \frac{w}{R} \cos^2 \lambda, \tag{7.83}$$

$$\varepsilon_{\varrho\lambda} = \frac{1}{\varrho} \cdot \frac{\partial u_\varrho}{\partial \lambda} + \frac{\partial u_\lambda}{\partial \varrho} - \frac{u_\lambda}{\varrho} + \frac{2}{R} w \sin \lambda \cos \lambda. \tag{7.84}$$

Multiplying (7.83) by ϱ and then differentiating with respect to ϱ, we obtain

$$\frac{\partial}{\partial \varrho}(\varrho \varepsilon_\lambda) = \frac{\partial^2 u_\lambda}{\partial \varrho \partial \lambda} + \frac{\partial u_\varrho}{\partial \varrho} + \frac{1}{R}\left(w + \varrho \frac{\partial w}{\partial \varrho}\right) \cos^2 \lambda. \tag{7.85}$$

Subtracting (7.85) from (7.82), we find

$$\varepsilon_\varrho - \frac{\partial}{\partial \varrho}(\varrho \varepsilon_\lambda) = \frac{w}{R} \sin^2 \lambda - \frac{\partial^2 u_\lambda}{\partial \varrho \partial \lambda} - \frac{1}{R}\left(w + \varrho \frac{\partial w}{\partial \varrho}\right) \cos^2 \lambda. \tag{7.86}$$

Differentiation of (7.84) with respect to λ yields

$$\frac{\partial}{\partial \lambda} \varepsilon_{\varrho\lambda} = -\frac{1}{\varrho} \cdot \frac{\partial}{\partial \lambda}\left(u_\lambda - \frac{\partial u_\varrho}{\partial \lambda}\right) + \frac{\partial^2 u_\varrho}{\partial \varrho \partial \lambda} + \frac{1}{R} \cdot \frac{\partial}{\partial \lambda}(w \sin 2\lambda). \tag{7.87}$$

Adding (7.86) and (7.87), we obtain

$$\frac{1}{\varrho}\cdot\frac{\partial}{\partial\lambda}\left(u_\lambda-\frac{\partial u_\varrho}{\partial\lambda}\right)=\varepsilon_\lambda-\varepsilon_\varrho+\frac{1}{R}w\cos 2\lambda+\frac{1}{R}\cdot\frac{\partial w}{\partial\lambda}\sin 2\lambda-$$
$$-\frac{\varrho}{R}\cdot\frac{\partial w}{\partial\varrho}\cos^2\lambda+\varrho\frac{\partial e_\lambda}{\partial\varrho}-\frac{\partial}{\partial\lambda}e_{\varrho\lambda}. \tag{7.88}$$

In addition, by (7.83)

$$\frac{1}{\varrho}\left(\frac{\partial u_\lambda}{\partial\lambda}+u_\varrho\right)=\varepsilon_\lambda-\frac{w}{R}\cos^2\lambda. \tag{7.89}$$

Before substituting (7.88) and (7.89) in (7.42), we shall insert the stress function σ. By Hooke's law,

$$\varepsilon_\varrho=\frac{1}{Eh}(S_\varrho-\nu S_\lambda),\quad \varepsilon_\lambda=\frac{1}{Eh}(S_\lambda-\nu S_\varrho),$$
$$e_{\varrho\lambda}=\frac{2(1+\nu)}{Eh}S_{\varrho\lambda}. \tag{7.90}$$

Substituting (7.90) in (7.89) and (7.88), we obtain, respectively,

$$\frac{1}{\varrho}\left(\frac{\partial u_\lambda}{\partial\lambda}+u_\varrho\right)=\frac{1}{Eh}(S_\lambda-\nu S_\varrho)-\frac{1}{R}w\cos^2\lambda, \tag{7.91}$$

$$\left.\begin{aligned}\frac{1}{\varrho}\cdot\frac{\partial}{\partial\lambda}\left(u_\lambda-\frac{\partial u_\varrho}{\partial\lambda}\right)&=\frac{1+\nu}{Eh}(S_\lambda-S_\varrho)+\frac{\varrho}{Eh}\cdot\frac{\partial}{\partial\varrho}(S_\lambda-\nu S_\varrho)-\\&-\frac{2(1+\nu)}{Eh}\cdot\frac{\partial}{\partial\lambda}S_{\varrho\lambda}+\frac{w}{R}\cos 2\lambda+\frac{1}{R}\cdot\frac{\partial w}{\partial\lambda}\sin 2\lambda-\frac{\varrho}{R}\frac{\partial w}{\partial\varrho}\cos^2\lambda.\end{aligned}\right\} \tag{7.92}$$

Substituting in the right-hand sides of (7.91) and (7.92) for S_ϱ, S_λ, $S_{\varrho\lambda}$, and w from (7.74) and (7.77), we obtain

$$\left.\begin{aligned}\frac{1}{\varrho}\left(\frac{\partial u_\lambda}{\partial\lambda}+u_\varrho\right)&=\frac{1-\nu}{2E}(p+q)-\frac{1+\nu}{2E}(p-q)\cos 2\lambda-\\&-\frac{1}{R}\operatorname{Re}\sigma\cdot\cos^2\lambda+\frac{\alpha}{h^2E}D\left[\frac{\nu}{\varrho}\cdot\frac{\partial}{\partial\varrho}+\frac{\nu}{\varrho^2}\cdot\frac{\partial^2}{\partial\lambda^2}-\frac{\partial^2}{\partial\varrho^2}\right]\operatorname{Im}\sigma,\end{aligned}\right\} \tag{7.93}$$

$$\left.\begin{aligned}\frac{1}{\varrho}\cdot\frac{\partial}{\partial\lambda}\left(u_\lambda-\frac{\partial u_\varrho}{\partial\lambda}\right)&=\frac{1+\nu}{E}(p-q)\cos 2\lambda+\frac{\alpha D}{h^2E}\left[\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho}+\right.\\&+\frac{1}{\varrho^2}(3+\nu)\frac{\partial^2}{\partial\lambda^2}-\frac{\partial^2}{\partial\varrho^2}-(2+\nu)\frac{1}{\varrho}\cdot\frac{\partial^3}{\partial\lambda^2\partial\varrho}-\varrho\frac{\partial^3}{\partial\varrho^3}\left.\right]\operatorname{Im}\sigma+\\&+\frac{1}{R}\left[\cos 2\lambda+\sin 2\lambda\cdot\frac{\partial}{\partial\lambda}-\varrho\cos^2\lambda\frac{\partial}{\partial\varrho}\right]\operatorname{Re}\sigma,\end{aligned}\right\} \tag{7.94}$$

where

$$\alpha=\sqrt{12(1-\nu^2)}. \tag{7.95}$$

Substitution of (7.93), (7.94), (7.74), and (7.75) in (7.42), taking (7.79) into account, yields the following boundary conditions at $\varrho = \varrho_0$ for the complex stress function σ:

$$\left\{\frac{a^2}{h^2}\varrho_0^2\left(\frac{1}{\varrho}\cdot\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\cdot\frac{\partial^2}{\partial\lambda^2}\right)+\delta_4\left[\frac{\nu}{\varrho^2}\cdot\frac{\partial^2}{\partial\lambda^2}-\frac{\partial^2}{\partial\varrho^2}+\frac{\nu}{\varrho}\cdot\frac{\partial}{\partial\varrho}\right]+\right.$$
$$+\delta_3\left[\frac{2+\nu}{\varrho}\cdot\frac{\partial^3}{\partial\lambda^2\partial\varrho}+\varrho\frac{\partial^3}{\partial\varrho^3\partial\lambda^2}-\frac{3+\nu}{\varrho^2}\cdot\frac{\partial^4}{\partial\lambda^4}+\frac{\partial^4}{\partial\varrho^2\partial\lambda^2}-\right.$$
$$\left.\left.-\frac{1}{\varrho}\cdot\frac{\partial^3}{\partial\lambda^2\partial\varrho}\right]\right\}\operatorname{Im}\sigma-8\beta^2\left\{\cos 2\lambda\left[\delta_3\left(5\frac{\partial^2}{\partial\lambda^2}+2\varrho\frac{\partial}{\partial\varrho}-\frac{\varrho}{2}\cdot\frac{\partial^3}{\partial\lambda^2\partial\varrho}-4\right)+\right.\right.$$
$$\left.+\frac{1}{2}\delta_4\right]+\delta_3\sin 2\lambda\left[\frac{\partial^3}{\partial\lambda^3}-8\frac{\partial}{\partial\lambda}+2\varrho\frac{\partial^2}{\partial\varrho\partial\lambda}\right]+$$
$$\left.+\frac{1}{2}\delta_4-\frac{\varrho^2}{2}\delta_2\frac{\partial^3}{\partial\lambda^2\partial\varrho}\right\}\operatorname{Re}\sigma=\left[\frac{aq_0^2}{2D}-\frac{a\delta_4(1-\nu)}{2Eh}\right](p+q)+$$
$$+\left[\frac{aq_0^2}{2D}-\frac{4a}{Eh}\delta_3(1+\nu)+\frac{a\delta_4}{2Eh}(1+\nu)\right](p-q)\cos 2\lambda,$$

$$\left\{\frac{a^2}{h^2}\varrho_0^2\left[\frac{\partial}{\partial\varrho}\left(\frac{1}{\varrho}\cdot\frac{\partial}{\partial\lambda}\right)\right]+\delta_4\left[\frac{\nu}{\varrho}\cdot\frac{\partial^2}{\partial\varrho\partial\lambda}+\frac{\nu}{\varrho^2}\cdot\frac{\partial^3}{\partial\lambda^3}-\frac{\partial^3}{\partial\varrho^2\partial\lambda}\right]+\right. \tag{7.96}$$
$$+\delta_3\left[\frac{1}{\varrho}\cdot\frac{\partial^3}{\partial\varrho\partial\lambda}+\frac{3+\nu}{\varrho^2}\cdot\frac{\partial^3}{\partial\lambda^3}-\frac{\partial^3}{\partial\varrho^2\partial\lambda}-\frac{2+\nu}{\varrho}\cdot\frac{\partial^4}{\partial\lambda^3\partial\varrho}-\right.$$
$$\left.\left.-\varrho\frac{\partial^4}{\partial\varrho^3\partial\lambda}\right]\right\}\operatorname{Im}\sigma-8\beta^2\left\{\cos 2\lambda\left[\delta_3\left(3\frac{\partial}{\partial\lambda}-\frac{\varrho}{2}\cdot\frac{\partial^3}{\partial\varrho\partial\lambda}\right)-\frac{1}{2}\delta_4\frac{\partial}{\partial\lambda}\right]+\right.$$
$$\left.+\sin 2\lambda\left[\delta_3\left(\frac{\partial^2}{\partial\lambda^2}+\varrho\frac{\partial}{\partial\varrho}-2\right)+\delta_4\right]-\frac{1}{2}\delta_4\frac{\partial}{\partial\lambda}\right\}\operatorname{Re}\sigma=$$
$$=\left[\frac{aq_0^2}{2D}+\frac{2a}{Eh}(1+\nu)\delta_3-\frac{a}{Eh}(1+\nu)\delta_4\right](p-q)\sin 2\lambda,$$

$$\frac{1}{\varrho_0}(\nu-\delta_1-\delta_2)\frac{\partial^2 w}{\partial\lambda^2}+(\nu-\delta_1)\frac{\partial w}{\partial\varrho}+\varrho_0\frac{\partial^2 w}{\partial\varrho^2}+\delta_2\frac{\partial^3 w}{\partial\varrho\partial\lambda^2}=0,$$

$$\varrho_0\frac{\partial^3 w}{\partial\varrho^3}+\frac{\partial^2 w}{\partial\varrho^2}-\frac{1}{\varrho_0}\cdot\frac{\partial w}{\partial\varrho}+\frac{1}{\varrho_0}(2-\nu+\delta_1+\delta_2)\frac{\partial^3 w}{\partial\varrho\partial\lambda^2}-$$
$$-\frac{1}{\varrho_0^2}(3-\nu+\delta_2)\frac{\partial^2 w}{\partial\lambda^2}+\frac{1}{\varrho_0^2}\delta_1\frac{\partial^4 w}{\partial\lambda^4}-\frac{p_0\varrho_0^2}{2D}=0,$$

where

$$\delta_1=\frac{A}{\varrho_0 D},\quad \delta_2=\frac{C}{\varrho_0 D},\quad \delta_3=\frac{B}{\varrho_0 D}\quad \delta_4=\frac{E_1 F\varrho_0}{D}. \tag{7.97}$$

We now substitute in (7.96) from (7.76) and (7.77) for $\operatorname{Im}\sigma$ and w, and equate the coefficients of β^0 and β^2 on both sides of each condition (7.96). (This is done separately for the free terms and for the coefficients of $\sin 2\lambda$ and $\cos 2\lambda$ or $\cos 4\lambda$, respectively.) We obtain a system of 12 algebraic equations, from which we find

$$A_0=\frac{a\pi\varrho_0^2}{4Eh}(p+q)\frac{(1+\nu)a_*-(1-\nu)\delta_4}{(1+\nu)a_*+(1+\nu)\delta_4}, \tag{7.98}$$

$$
\left.\begin{aligned}
C_0 &= \frac{\alpha\pi\varrho_0^2}{2Eh\alpha_0}(p-q)(1+\nu)[\alpha_*^2+\alpha_*(\delta_4+4\delta_3)-12\delta_3\delta_4],\\
F_1 &= -\frac{\alpha\pi\varrho_0^4}{8Eh\alpha_0}(p-q)[(1+\nu)\alpha_*^2+4\alpha_*(3+\nu)\delta_3-\alpha_*(1-\nu)\delta_4-\\
&\qquad -12(1+\nu)\delta_3\delta_4],\\
A_1 &= \frac{\pi\varrho_0^2}{2}\left(A_0-\frac{1}{2}C_0\right)\frac{\alpha_*(1+\nu)-(1-\nu)\delta_4}{(1+\nu)(\alpha_*+\delta_4)},\\
C_1 &= \frac{\pi\varrho_0^2}{2\alpha_0}(C_0-A_0)(\alpha_*^2+4\alpha_*\delta_3+\alpha_*\delta_4-12\delta_3\delta_4-12\delta_3\delta_4)(1+\nu),\\
F_2 &= -\frac{\pi\varrho_0^4}{8\alpha_0}(C_0-A_0)[\alpha_*^2(1+\nu)+4\alpha_*(3+\nu)\delta_3-\alpha_*(1-\nu)\delta_4-\\
&\qquad -12(1+\nu)\delta_3\delta_4],\\
A_2 &= \frac{\alpha\pi\varrho_0^2}{2Eh^2}p_0R-\left(A_0-\frac{1}{2}C_0\right),\\
B_1 &= \frac{\varrho_0^2}{2(1-\nu+\delta_1)}\left\{(1+\nu-\delta_1)\left[\left(2\gamma'-\frac{1}{2}\right)C_0+2(1-2\gamma')A_0\right]-\right.\\
&\qquad \left.-(3+\nu-\delta_1)\frac{\alpha\pi\varrho_0^2 q}{Eh}\right\},\\
D_1 &= \frac{\varrho_0^2}{2\alpha_9}(\alpha_1A_0-\alpha_2C_0),\quad H_2=\frac{\varrho_0^2}{48\alpha_{10}}(\alpha_5\varrho_0^2C_0-\alpha_6F_1),\\
E_2 &= -\frac{\varrho_0^4}{12\alpha_9}(\alpha_3A_0+\alpha_4C_0)-2H_2,\quad M_3=\frac{\varrho_0^6}{1440\alpha_{10}}(\alpha_7\varrho_0^2C_0+\alpha_8F_1),
\end{aligned}\right\}
\qquad (7.98)\ \text{cont'd}
$$

where

$$
\left.\begin{aligned}
\alpha &= \sqrt{12(1-\nu^2)},\quad \alpha_* = 12(1-\nu)\frac{\varrho_0^2}{h^2},\\
\alpha_0 &= (1+\nu)\alpha_*^2+\alpha_*(3+\nu)(4\delta_3+\delta_4)+12(3-\nu)\delta_3\delta_4\\
\alpha_1 &= (1-\nu)[4\gamma'(1-\nu)+(5-\nu)]+[(1-\nu)(4\gamma'+3)+2(3+\nu)]\delta_1+\\
&\quad +[2(1-\nu)(2\gamma'+1)+(3+\nu)]\delta_2-3\delta_1\delta_2(1+4\gamma'),\\
\alpha_2 &= (1-\nu)\left[4\gamma'(1-\nu)+\frac{1}{3}(13-\nu)\right]+\\
&\quad +\left[4(1-\nu)\gamma'+\frac{1}{3}(25-\nu)\right]\delta_1+\\
&\quad +4(1-\nu)\gamma'+\frac{1}{3}(13-\nu)\Big]\delta_2-(1+12\gamma')\delta_1\delta_2\\
\alpha_3 &= -6\gamma'(1-\nu)^2-(9-4\nu+3\nu^2)+[6\gamma'(\nu-3)+(3\nu-13)]\delta_1+\\
&\quad +[6\gamma'(1+\nu)+(3\nu+5)]\delta_2+9(1+2\gamma')\delta_1\delta_2\\
\alpha_4 &= 6\gamma'(1-\nu)^2+2(\nu^2-\nu+4)+[6\gamma'(3-\nu)+2(5-\nu)]\delta_1-\\
&\quad -[6\gamma'(1+\nu)+2(2+\nu)]\delta_2-6(1+3\gamma')\delta_1\delta_2\\
\alpha_5 &= (1-\nu)(1-3\nu)-(5+3\nu)\delta_1+(1-3\nu)\delta_2-45\delta_1\delta_2\\
\alpha_6 &= 4[3(1-\nu^2)+(15+3\nu)\delta_1+3(3+\nu)\delta_2+45\delta_1\delta_2\\
\alpha_7 &= (19\nu-3-10\nu^2)-(9-10\nu)\delta_1+(6+10\nu)\delta_2+150\delta_1\delta_2\\
\alpha_8 &= 6[(6\nu-5\nu^2+3)+(\delta_1+\delta_2)(9+5\nu)+75\delta_1\delta_2]\\
\alpha_9 &= (1-\nu)(3+\nu)+(5+\nu)(\delta_1+\delta_2)+3\delta_1\delta_2\\
\alpha_{10} &= (1-\nu)(3+\nu)+(9+\nu)(\delta_1+\delta_2)+15\delta_1\delta_2
\end{aligned}\right\}.
\qquad (7.99)
$$

Setting $\delta_1=\delta_2=\delta_3=\delta_4=0$, in (7.98) and (7.99), we obtain the solution for the case considered by Lur'e /68/, when the edge of the hole is not reinforced:*

$$\left.\begin{aligned} A_0&=\frac{a\pi\varrho_0^2}{4Eh}(p+q), \quad C_0=\frac{a\pi\varrho_0^2}{2Eh}(p-q)\\ A_1&=\frac{\pi\varrho_0^2}{2}\left(A_0-\frac{1}{2}C_0\right), \quad C_1=\frac{\pi\varrho_0^2}{2}(C_0-A_0)\\ F_1&=-\frac{a\pi\varrho_0^4}{8Eh}(p-q), \quad F_2=-\frac{\pi\varrho_0^4}{8}(C_0-A_0)\end{aligned}\right\}, \tag{7.100}$$

$$\left.\begin{aligned} &A_2=0,\\ &D_1=\frac{\varrho_0^2}{2(3+\nu)}\left\{A_0[4\gamma'(1-\nu)+(5-\nu)]-\right.\\ &\qquad\left.-C_0[4\gamma'(1-\nu)+\frac{1}{3}(13-\nu)]\right\}\\ &B_1=\frac{\varrho_0^2}{2(1-\nu)}\left\{(1+\nu)\left[\left(2\gamma'-\frac{1}{2}\right)C_0+2(1-2\gamma')A_0\right]-\right.\\ &\qquad\left.-(3+\nu)\frac{a\pi\varrho_0^2}{Eh}\cdot\frac{P_0R}{h}\right\}\\ &E_2+2H_2=\frac{\varrho_0^4}{12(1-\nu)(3+\nu)}\left\{A_0[6\gamma'(1-\nu)^2+\right.\\ &\quad\left.+(9-4\nu+3\nu^2)]-C_0[6\gamma'(1-\nu)^2+2(\nu^2-\nu+4)]\right\}\\ &H_2=\frac{\varrho_0^4}{48(3+\nu)}\left[C_0(1-3\nu)-12(1+\nu)F_1\cdot\frac{1}{\varrho_0^2}\right]\\ &M_3=\frac{\varrho_0^6}{1440(1-\nu)(3+\nu)}\left[C_0(19\nu-3-10\nu^2)+\right.\\ &\qquad\left.+6(6\nu-5\nu^2+3)F_1\cdot\frac{1}{\varrho_0^2}\right]\end{aligned}\right\}. \tag{7.100a}$$

In the limit $\delta_1=\delta_2=\delta_3=\delta_4=\infty$ we obtain from (7.98) and (7.99) the coefficients for the case when a perfectly rigid rib or washer is welded to the edge of the hole:

$$\left.\begin{aligned} A_0&=-\frac{a\pi\varrho_0^2(1-\nu)}{4Eh(1+\nu)}(p+q), \qquad C_0=-\frac{a\pi\varrho_0^2(1+\nu)}{2Eh(3-\nu)}(p-q)\\ F_1&=\frac{a\pi\varrho_0^4(1+\nu)}{8Eh(3-\nu)}(p-q), \qquad A_1=-\frac{\pi\varrho_0^2(1-\nu)}{2(1+\nu)}\left(A_0-\frac{1}{2}C_0\right)\\ C_1&=-\frac{\pi\varrho_0^2(1+\nu)}{2(3-\nu)}(C_0-A_0), \qquad F_2=\frac{\pi\varrho_0^4(1+\nu)}{8(3-\nu)}(-A_0+C_0)\\ A_2&=\frac{a\pi\varrho_0^2}{2Eh}\cdot\frac{p_0R}{h}-\left(A_0-\frac{1}{2}C_0\right)\\ B_1&=\frac{\varrho_0^2}{2}\left[C_0\left(\frac{1}{2}-2\gamma'\right)+2A_0(2\gamma'-1)+\frac{a\pi\varrho_0^2}{Eh}\cdot\frac{p_0R}{h}\right]\\ D_1&=\frac{\varrho_0^2}{6}[C_0(12\gamma'+1)-3A_0(4\gamma'+1)]\end{aligned}\right\}, \tag{7.101}$$

* An inaccuracy exists in equations (4.9.22) and (4.9.15) of Lur'e's solution /68/.

$$E_2 + 2H_2 = \frac{\varrho_0^4}{12}[2C_0(3\gamma' + 1) - 3A_0(1 + 2\gamma')]$$
$$H_2 = -\frac{\varrho_0^4}{16}\left(C_0 + \frac{4}{\varrho_0^2}F_1\right), \quad M_3 = \frac{\varrho_0^4}{144}(\varrho_0^2 C_0 + 3F_1) \Bigg\}. \tag{7.101a}$$

This case was considered in /95/ and /176/, but the solutions obtained contain some inaccuracies. In the former solution the boundary condition $w = c$ at $\varrho = \varrho_0$ is not satisfied (c is a given constant), while in the latter the equilibrium conditions for a rigid rib are not satisfied. (Integration along Γ of the generalized shearing forces Q_0^* does not yield $\pi\varrho_0^2 p_0$.) Similar inaccuracies are also contained in the solutions obtained in /96/ and /97/, which deal with the problem of stresses in a cylindrical shell with a circular hole whose edge is reinforced by a wide elastic rib. Furthermore, the equilibrium conditions are not satisfied for the elastic stiffening rib.

TABLE 31

δ	k_1	k_2	k_3	k_4
0	0.000	0.000	2.500	2.920
0.5	0.309	0.259	1.976	1.661
1	0.502	0.322	1.680	1.081
2	0.736	0.300	1.344	0.551
5	1.031	0.119	0.942	0.110
10	1.190	—0.059	0.726	—0.035
20	1.279	—0.204	0.585	—0.093
100	1.253	—0.374	0.420	—0.126
380	1.064	—0.372	0.331	—0.116
∞	0.783	—0.213	0.235	—0.064

TABLE 32

δ	0	0.5	1	2	5	10	20	100	380	∞
$\frac{1}{q}\sigma_{\varrho_1}^P$	0.000	0.102	0.213	0.390	0.668	0.853	0.985	1.157	1.254	1.364
$\frac{1}{q}\sigma_{\lambda_2}^P$	0.080	0.115	0.153	0.194	0.248	0.284	0.311	0.350	0.377	0.409

We calculated the normal stresses $\sigma_{\varrho 1} = \frac{1}{h}S_\varrho$ and $\sigma_{\lambda 1} = \frac{1}{h}S_\lambda$ ((7.74)) at the point $P(\varrho_0, 0)$ on the middle surface of the shell for the case of a stiffening rib having a rectangular section with sides b and h_1. These stresses are

$$\sigma_{\varrho 1}^P = q\left(k_1 + k_2\frac{\varrho_0^2}{Rh}\right), \quad \sigma_{\lambda 1}^P = q\left(k_3 + k_4\frac{\varrho_0^2}{Rh}\right), \tag{7.102}$$

where the first terms correspond to plane biaxial tension.

The coefficients k_1, k_2, k_3, k_4 are given in Table 31 for various values of δ, with $b = 0.1\varrho_0$, $\nu = 0.3$, $\varrho_0 = 15h$, $h_1 = 3h$, $h_1 = 2b$, where $\sigma_{\varrho 2}^P$ and are, respectively, the moduli of elasticity of rib and shell.

The normal stresses $\sigma_{\varphi 2}^{P}$ and $\sigma_{\lambda 2}^{P}$ induced at $P(\varrho_0, 0)$ on the outer surface of the shell by the bending moments M_φ and M_λ ((7.77)), were calculated for

$$\varrho_0 = \frac{1}{2}\sqrt{Rh}.$$

The results are given in Table 32.

The total stresses

$$\sigma_\varphi = \sigma_{\varphi 1} + \sigma_{\varphi 2}, \qquad \sigma_\lambda = \sigma_{\lambda 1} + \sigma_{\lambda 2}$$

at points $P(\varrho_0, 0)$ and $N\left(\varrho_0, \frac{\pi}{2}\right)$ on the outer surface of the shell are given in Figure 103 as curves of the most characteristic magnitudes $\frac{1}{q}\sigma_\lambda^P$ (curve 1), $\frac{1}{q}\sigma_\varphi^N$ (curve 2), and $\frac{1}{q}\sigma_\varphi^P$ (curve 3), where

$$\varkappa = \frac{\delta}{\delta + 20}.$$

For $\varkappa \approx 0.14$, i.e., $\delta \approx 3.25$, we have $\sigma_\varphi^P = \sigma_\lambda^P \cong 1.45q$. The stress-concentration coefficient for $P(\varrho_0, 0)$ has thus been reduced by a factor of $\frac{3.31}{1.45} = 2.28$ through reinforcing the edge of the hole by an elastic rib.

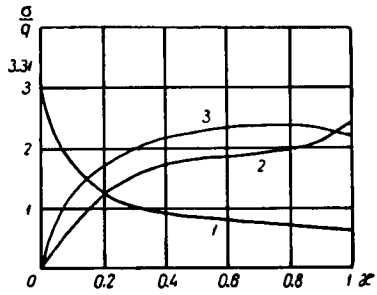


FIGURE 103.

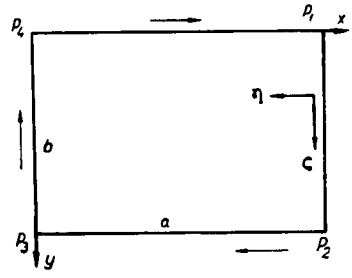


FIGURE 104.

§ 6. SHALLOW SPHERICAL SHELL, RECTANGULAR IN THE PLAN, WITH REINFORCED EDGES

Consider an elastic shallow spherical shell whose projection in plan is a rectangle with sides a and b (Figure 104). The shell edges are reinforced by thin elastic ribs possessing torsional and tensile rigidity and flexural rigidity in two planes. The stiffening ribs are welded together at the corners of the rectangle. The reinforced edges of the shell are hinged to a rigid foundation. A uniformly distributed load $q = \text{const.}$ acts on the upper surface in a downward direction.

The complex stress-displacement function (cf., for instance, /88/, § 17, Chapter I)

$$\psi(x, y) = w + i\varphi = w - \frac{\sqrt{12(1-\nu^2)}}{Eh^2} i\Phi, \tag{7.103}$$

through which all stress and strain components of the shell are expressed, must in the case considered satisfy the differential equation

$$\Delta\Delta\psi - 2ik^2\Delta\psi = \frac{q}{D} \tag{7.104}$$

and certain boundary conditions.

Here, w is the deflection toward the center of curvature of the shell, Φ is the stress function corresponding to a plane state of stress, h is the shell thickness, $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity of the shell, R is the radius of curvature of the shell, Δ is the Laplacian operator,

$$k^2 = \frac{\sqrt{3(1-\nu^2)}}{Rh}.$$

Since the structure, the load, and the boundary conditions are symmetrical with respect to the axes $x = \frac{a}{2}$, $y = \frac{b}{2}$, we can represent the general integral ψ of the nonhomogeneous differential equation (7.104) in the form /47/

$$\begin{aligned}\psi(x, y) &= \psi_0(x, y) + \psi_1(x, y) = \psi_2(x, y) + \alpha^*\omega_x + \beta^*\omega_y = \\ &= \psi_0(x, y) + \alpha^*\omega_x + \beta^*\omega_y + \sum_{n=1,3,\ldots}\left(\frac{b}{n\pi}\right)^2 [A_n\zeta_{2,n}(\beta, \xi) + B_n t_n^2\zeta_{0,n}(\beta, \xi)]\times \\ &\times \sin n\pi\eta + \sum_{m=1,3,\ldots}\left(\frac{a}{m\pi}\right)^2 [C_m\zeta_{2,m}(\alpha, \eta) + D_m s_m^2\zeta_{0,m}(\alpha, \eta)]\sin m\pi\xi,\end{aligned} \tag{7.105}$$

where $\psi_0(x, y)$ is any particular integral of (7.104), α^*, β^*, A_n, B_n, C_m, D_m are complex integration constants, to be determined from the boundary conditions, $\xi = \frac{x}{a}$, $\eta = \frac{y}{b}$, $\alpha = \frac{m\pi b}{a}$, $\beta = \frac{n\pi a}{b}$,

$$\left.\begin{aligned}\omega_x &= \frac{i}{2k^2}\cdot\frac{\operatorname{ch}\frac{(1+i)ka(1-2\xi)}{2}}{\operatorname{ch}\frac{(1+i)ka}{2}}, \quad \omega_y = \frac{i}{2k^2}\cdot\frac{\operatorname{ch}\frac{(1+i)kb(1-2\eta)}{2}}{\operatorname{ch}\frac{(1+i)kb}{2}}, \\ \zeta_{0,n}(\beta, \xi) &= \frac{1}{t_n^2-1}\left[\frac{\operatorname{ch}\frac{\beta(1-2\xi)}{2}}{\operatorname{ch}\frac{\beta}{2}} - \frac{\operatorname{ch}\frac{t_n\beta(1-2\xi)}{2}}{t_n^2\operatorname{ch}\frac{t_n\beta}{2}}\right] \\ \zeta_{2,n}(\beta, \xi) &= \frac{1}{t_n^2-1}\left[\frac{\operatorname{ch}\frac{\beta(1-2\xi)}{2}}{\operatorname{ch}\frac{\beta}{2}} - \frac{\operatorname{ch}\frac{t_n\beta(1-2\xi)}{2}}{\operatorname{ch}\frac{t_n\beta}{2}}\right]\end{aligned}\right\} \tag{7.106}$$

$$\left.\begin{aligned}
\zeta_{0,m}(\alpha,\eta) &= \frac{1}{s_m^2-1}\left[\frac{\operatorname{ch}\frac{\alpha(1-2\eta)}{2}}{\operatorname{ch}\frac{\alpha}{2}} - \frac{\operatorname{ch}\frac{s_m\alpha(1-2\eta)}{2}}{s_m^2\operatorname{ch}\frac{s_m\alpha}{2}}\right]\\
\zeta_{2,m}(\alpha,\eta) &= \frac{1}{s_m^2-1}\left[\frac{\operatorname{ch}\frac{\alpha(1-2\eta)}{2}}{\operatorname{ch}\frac{\alpha}{2}} - \frac{\operatorname{ch}\frac{s_m\alpha(1-2\eta)}{2}}{\operatorname{ch}\frac{s_m\alpha}{2}}\right]\\
s_m &= \frac{1}{m\pi}\sqrt{m^2\pi^2+2ik^2a^2}\,,\quad t_n = \frac{1}{n\pi}\sqrt{n^2\pi^2+2ik^2b^2}
\end{aligned}\right\}. \qquad (7.106)\ \text{cont'd}$$

We take as particular integral $\psi_0(x,y)$ the solution of the problem of bending for the same shell without stiffening ribs, subjected to a load $q = \text{const}$. and hinged at the edges to an inextensible support /47/:

$$\psi_0(x,y) = \sum_{m=1}^{\infty}\frac{2qa^4}{Dm^5\pi^5}[1+(-1)^{m+1}]\left\{\frac{1}{s_m^2}-\right.$$
$$\left.-\frac{1}{s_m^2-1}\left[\frac{\operatorname{ch}\frac{\alpha(1-2\eta)}{2}}{\operatorname{ch}\frac{\alpha}{2}} - \frac{\operatorname{ch}\frac{s_m\alpha(1-2\eta)}{2}}{s_m^2\operatorname{ch}\frac{s_m\alpha}{2}}\right]\right\}\sin m\pi\xi. \qquad (7.107)$$

The function $\psi_1(x, y)$ defines the additional state of stress arising because the actual support conditions at the shell edges differ from those of a simple support.

The boundary conditions for the reinforced edges of the shell are obtained from (7.49) for $\varrho = \infty$. Setting $\frac{1}{R} = 0$ to simplify them, we obtain finally:

at $x = \begin{cases} a \\ 0 \end{cases}$

$$\left.\begin{aligned}
\beta'\left[\frac{\partial^5\Phi}{\partial x^3\partial y^2}+(\nu+2)\frac{\partial^5\Phi}{\partial x\partial y^4}\right] &= \pm Eh\frac{\partial^2\Phi}{\partial y^2}\\
E_1F\left(\frac{\partial^3\Phi}{\partial x^2\partial y}-\nu\frac{\partial^3\Phi}{\partial y^3}\right) &= \mp Eh\frac{\partial^2\Phi}{\partial x\partial y}\\
\mp C\frac{\partial^3 w}{\partial x\partial y^2}+D\frac{\partial^2 w}{\partial x^2} &= 0\\
w &= 0
\end{aligned}\right\}; \qquad (7.108)$$

at $y = \begin{cases} b \\ 0 \end{cases}$

$$\left.\begin{aligned}
\beta'\left[\frac{\partial^5\Phi}{\partial x^2\partial y^3}+(\nu+2)\frac{\partial^5\Phi}{\partial x^4\partial y}\right] &= \pm Eh\frac{\partial^2\Phi}{\partial x^2}\\
E_1F\left(\frac{\partial^3\Phi}{\partial x\partial y^2}-\nu\frac{\partial^3\Phi}{\partial x^3}\right) &= \mp Eh\frac{\partial^2\Phi}{\partial x\partial y}\\
\mp C\frac{\partial^3 w}{\partial x^2\partial y}+D\frac{\partial^2 w}{\partial y^2} &= 0\\
w &= 0
\end{aligned}\right\}. \qquad (7.109)$$

The upper sign corresponds to the edge $x = a$ or $y = b$, and the lower sign to the edge $x = 0$ or $y = 0$.

If we set $l_1 = 1$ and $m_1 = 0$ in the expression (7.30) for β, so that one of the principal axes of inertia of the rib section lies in the middle surface of the shell, we obtain $\beta' = A$. In this case (7.108) and (7.109) coincide with the corresponding boundary conditions given in /47/.

In addition to the above boundary conditions certain special conditions must also be satisfied at the corners. These are the equality of action and reaction, i.e., the equality of the internal forces and moments acting in adjacent sections of the two ribs meeting at the corner. It suffices to satisfy these conditions at one corner, for instance at $x = y = 0$, since, because of the symmetry with respect to the axes $\xi = \eta = \frac{1}{2}$, similar conditions will then be satisfied automatically at the other three corners.

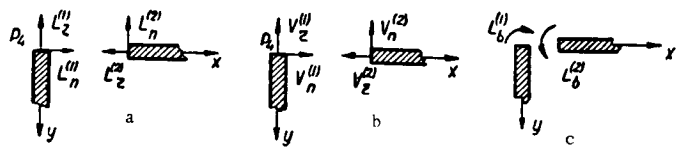


FIGURE 105.

These special conditions are:

1) The twisting moment $L_\xi^{(1)}$ acting on section P_4 of rib P_3P_4 is equal to the bending moment $L_n^{(2)}$ acting on section P_4 of rib P_4P_1, and vice versa:

$$L_\xi^{(2)} = L_n^{(1)}, \qquad L_\xi^{(1)} = -L_n^{(2)}. \tag{7.110}$$

The positive directions of these vectors are shown in Figure 105,a.

2) The longitudinal force $V_\xi^{(1)}$ acting on section P_4 of rib P_3P_1 is equal to the shearing force $V_n^{(2)}$ acting on section P_4 of rib P_4P_1, and vice versa:

$$V_\xi^{(2)} = V_n^{(1)}; \qquad V_\xi^{(1)} = -V_n^{(2)}. \tag{7.111}$$

The positive direction of these forces is shown in Figure 105,b.

3) The bending moment $L_b^{(1)}$ acting on section P_4 of the rib P_3P_4 is equal to the bending moment $L_b^{(2)}$ acting on section P_4 of rib P_4P_1 (Figure 105,c):

$$L_b^{(1)} = L_b^{(2)}. \tag{7.112}$$

The moments $L_n^{(1)}$, $L_b^{(1)}$, $L_\xi^{(1)}$, entering in (7.110) to (7.112) are by (7.16) and (7.14) for $\omega_\xi = \omega_\eta = \omega_\zeta = 0$:

$$\left.\begin{aligned} L_b^{(1)} &= l_1 L_\xi^{(1)} + m_1 L_\eta^{(1)} = \beta' \frac{\partial^2 u}{\partial y^2} \\ L_n^{(1)} &= m_1 L_\xi^{(1)} - l_1 L_\eta^{(1)} = \gamma \frac{\partial^2 u}{\partial y^2} \\ L_\xi^{(1)} &= -C \frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\}. \tag{7.113}$$

Similarly,

$$L_\xi^{(2)} = \beta' \frac{\partial^3 v}{\partial x^3}, \qquad L_n^{(2)} = \gamma \frac{\partial^3 v}{\partial x^3}, \qquad L_\zeta^{(2)} = C \frac{\partial^3 w}{\partial x \partial y}, \tag{7.114}$$

where u and v are the displacement components in the x- and y- directions, respectively.

The shearing forces are by (7.6) and (7.17):

$$\left.\begin{aligned} V_n^{(1)} &= m_1 V_\xi^{(1)} - l_1 V_\eta^{(1)} = -\beta' \frac{\partial^3 u}{\partial y^3} \\ V_n^{(2)} &= m_1 V_\xi^{(2)} - l_1 V_\eta^{(2)} = \beta' \frac{\partial^3 v}{\partial x^3} \\ V_\zeta^{(1)} &= E_1 F \frac{\partial v}{\partial y}, \qquad V_\zeta^{(2)} = E_1 F \frac{\partial u}{\partial x} \end{aligned}\right\}. \tag{7.115}$$

Substitution of (7.113), (7.114), and (7.115) in (7.110), (7.111), and (7.112) yields:

at $x = y = 0$

$$\left.\begin{aligned} C \frac{\partial^2 w}{\partial x \partial y} &= +\gamma \frac{\partial^2 u}{\partial y^2}, \qquad C \frac{\partial^2 w}{\partial x \partial y} = +\gamma \frac{\partial^2 v}{\partial x^2} \\ -E_1 F \frac{\partial u}{\partial x} &= \beta' \frac{\partial^3 u}{\partial y^3}, \qquad -E_1 F \frac{\partial v}{\partial y} = \beta' \frac{\partial^3 v}{\partial x^3} \end{aligned}\right\}, \tag{7.116}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}. \tag{7.117}$$

Condition (7.117) is fulfilled automatically because of the first two conditions (7.116). There remain thus four conditions for determining the real and imaginary parts of the complex coefficients α^* and β^* appearing in (7.105).

Conditions (7.116) are expressed as follows through the stress function $\Phi(x, y)$:

$$\left.\begin{aligned} -C \frac{\partial^2 w}{\partial x \partial y} &= \frac{\gamma}{E} \left[\frac{\partial^3}{\partial x^3} \Phi + (2 + \nu) \frac{\partial^3 \Phi}{\partial x \partial y^2} \right] \\ -C \frac{\partial^2 w}{\partial x \partial y} &= \frac{\gamma}{E} \left[\frac{\partial^3 \Phi}{\partial y^3} + (2 + \nu) \frac{\partial^3 \Phi}{\partial y \partial x^2} \right] \\ E_1 F \left(\frac{\partial^2 \Phi}{\partial y^2} - \nu \frac{\partial^2 \Phi}{\partial x^2} \right) &= \beta' \left[\frac{\partial^4 \Phi}{\partial x^3 \partial y} + (2 + \nu) \frac{\partial^4 \Phi}{\partial x \partial y^3} \right] \\ E_1 F \left(\frac{\partial^2 \Phi}{\partial x^2} - \nu \frac{\partial^2 \Phi}{\partial y^2} \right) &= +\beta' \left[\frac{\partial^4 \Phi}{\partial x \partial y^3} + (2 + \nu) \frac{\partial^4 \Phi}{\partial x^3 \partial y} \right] \end{aligned}\right\}. \tag{7.118}$$

Substituting (7.105) in (7.118), we obtain a system of four real equations, from which we find

$$\alpha^* = \frac{\operatorname{ch} ka + \cos ka}{2(\operatorname{sh} ka + \sin ka)} f_1, \qquad \beta^* = \frac{\operatorname{ch} kb + \cos kb}{2(\operatorname{sh} kb + \sin kb)} f_2, \tag{7.119}$$

where

$$f_1 = -\frac{2\beta'(1-i)}{E_1F(1-\nu)}\,\mathrm{th}\,\frac{(1-i)\,ka}{2}\,\mathrm{Im}\left[2\frac{\partial^4\psi_2}{\partial y\partial x^3}+(1+\nu)\frac{\partial^4\psi_2}{\partial x\partial y^3}\right]_{x=y=0} -$$
$$-\frac{2}{k}\left\{\frac{2k^2}{\gamma}RC\,\mathrm{Re}\left(\frac{\partial^2\psi_2}{\partial x\partial y}\right)+\mathrm{Im}\left[\frac{\partial^3\psi_2}{\partial x^3}+(2+\nu)\frac{\partial^3\psi_2}{\partial x\partial y^2}\right]\right\}_{x=y=0},$$
$$f_2 = -\frac{2\beta'(1-i)}{E_1F(1-\nu)}\,\mathrm{th}\,\frac{(1-i)\,kb}{2}\,\mathrm{Im}\left[2\frac{\partial^4\psi_2}{\partial x\partial y^3}+(1+\nu)\frac{\partial^4\psi_2}{\partial y\partial x^3}\right]_{x=y=0} -$$
$$-\frac{2}{k}\left\{\frac{2k^2}{\gamma}RC\,\mathrm{Re}\left(\frac{\partial^2\psi_2}{\partial x\partial y}\right)+\mathrm{Im}\left[\frac{\partial^3\psi_2}{\partial y^3}+(2+\nu)\frac{\partial^3\psi_2}{\partial y\partial x^2}\right]\right\}_{x=y=0}$$

We have thus expressed α^* and β^* through the coefficients A_n, B_n, C_m, D_m entering in $\left(\frac{\partial^k\psi_2}{\partial x^l\partial y^{k-l}}\right)_{x=y=0}$ [cf. (7.105)]. The coefficients A_n, B_n, C_m, D_m are determined from (7.108) and (7.109). Substituting the functions $w(x, y)$ and $\Phi(x, y)$ ((7.103)) in these boundary conditions and using (7.105), we obtain two groups of equations expressing the conditions at $x = \begin{cases} a \\ 0 \end{cases}$ and $y = \begin{cases} b \\ 0 \end{cases}$, respectively. Omitting the lengthy calculations by which $\zeta_{j,n}(\beta, \xi)$, $\zeta_{l,m}(\alpha, \eta)$ $(j = 0, 1, 2 \ldots 6)$ and some other functions are expanded in trigonometric series, we obtain an infinite system of linear algebraic equations in the unknown coefficients A_n, B_n, C_m, D_m:

$$\mathrm{Im}\left\{A_n\left(\frac{n\pi}{b}\right)^3[\zeta_{5,n}(\beta, 0)-(\nu+2)\zeta_{3,n}(\beta, 0]+\right.$$
$$+B_nt_n^2\left[\left(\frac{n\pi}{b}\right)^3\zeta_{3,n}(\beta, 0)-(\nu+2)\left(\frac{n\pi}{b}\right)^3\zeta_{1,n}(\beta, 0)+\frac{Eh}{\beta'}\zeta_{0,n}(\beta, 0)\right]+$$
$$+\sum_m\left(\frac{m\pi}{a}\right)^3C_m\left[-\frac{4n^3}{\pi}+\frac{4\pi(\nu+2)}{a^4\alpha^2}ns_m^2m^4b^4+\right.$$
$$\left.+\frac{4\pi}{a^2\alpha^2}(\nu+2)(1+s_m^2)m^2n^3b^2\right]\frac{1}{\delta_{mn}}+\sum_m\left(\frac{m\pi}{a}\right)^3D_m\left[\frac{\alpha^2}{\pi^2}+(\nu+2)n^2\right]\frac{4ns_m^2}{\pi\delta_{mn}}+$$
$$\left.+\frac{4Eh}{\pi nt_n^2\beta'}\beta^*-\sum_m\frac{16qn}{Da\pi\delta_{mn}}\left[\frac{\alpha^2}{\pi^2}+n^2(\nu+2)\right]\right\}=0,$$
$$\mathrm{Im}\left\{A_n\left[\frac{n\pi}{b}\zeta_{4,n}(\beta, 0)-\frac{Eh}{E_1F}\zeta_{3,n}(\beta, 0)\right]+\right.$$
$$+B_nt_n^2\left[\frac{\nu n\pi}{b}\zeta_{0,n}(\beta, 0)-\frac{Eh}{E_1F}\zeta_{1,n}(\beta, 0)\right]-\frac{Eh}{E_1h}\sum_m\frac{4an^2}{\pi^2\delta_{mn}}C_m+$$
$$\left.+\frac{Eh}{E_1F}\sum_m\frac{4\alpha^3s_m^2}{\pi^4\delta_{mn}}D_m-\frac{8\nu k^2ib}{\pi^2n^2t_n^2}\beta^*-\frac{Eh}{E_1F}\cdot\frac{16qa^2}{D\pi^7}\sum_m\frac{\alpha^3}{m^3\delta_{mn}}\right\}=0,$$
$$\mathrm{Re}\left\{A_n\left[\frac{D}{C}\zeta_{4,n}(\beta, 0)-\frac{n\pi}{b}\zeta_{3,n}(\beta, 0)\right]-B_nt_n^2\frac{n\pi}{b}\zeta_{1,n}(\beta, 0)-\right.$$
$$\left.-\sum_m\frac{4mn^3}{a\delta_{mn}}C_m+\sum_m\frac{4m\alpha^2n}{a\pi^2\delta_{mn}}D_ms_m^2-\frac{D}{C}\cdot\frac{4}{n\pi}\alpha^*-\sum_m\frac{16qa\alpha^2n}{Dm^2\pi^3\delta_{mn}}\right\}=0,$$

$$\operatorname{Re}\left\{\left(\frac{b}{n\pi}\right)^3 B_n t_n^2 \zeta_{0,n}(\beta, 0) + \frac{2l}{\pi k^2 n}\left(\alpha^* + \frac{1}{t_n^2}\beta^*\right)\right\} = 0$$

$$(n = 1, 3, 5, \ldots),$$

$$\operatorname{Im}\left\{\sum_n \left(\frac{n\pi}{b}\right)^3 A_n\left[-\frac{m^3}{\pi} + \frac{(\nu+2)\pi}{b^4\beta^2} m t_n^2 n^4 a^4 + \right.\right.$$

$$+\frac{(\nu+2)\pi}{b^2\beta^3}(1+t_n^2)n^2m^3a^2\Big]\frac{4}{\Delta_{mn}} + \sum_n\left(\frac{n\pi}{b}\right)^3 B_n\left[\frac{b^2}{\pi^2}+(\nu+2)m^2\right]\frac{4mt_n^2}{\pi\Delta_{mn}} +$$

$$+ C_m\left(\frac{m\pi}{a}\right)^3[\zeta_{5,m}(\alpha, 0) - (\nu+2)\zeta_{3,m}(\alpha, 0)] +$$

$$+ D_m s_m^2\left[\left(\frac{m\pi}{a}\right)^3\zeta_{3,m}(\alpha, 0) - (\nu+2)\left(\frac{n\pi}{a}\right)^3\zeta_{1,m}(\alpha, 0) + \frac{Eh}{\beta'}\zeta_{0,m}(\alpha, 0)\right] +$$

$$+ \frac{4Eh}{\pi m s_m^2\beta'}\alpha^* - \frac{4q}{Da}\zeta_{3,m}(\alpha, 0) + (\nu+2)\frac{4q}{Da}\zeta_{1,m}(\alpha, 0) +$$

$$\left.+ \frac{4Ehqa^2}{Dm^3\pi^3\gamma}\left[\frac{1}{s_m^2} - \zeta_{0,m}(\alpha, 0)\right]\right\} = 0,$$

$$\operatorname{Im}\left\{C_m\left[\frac{m\pi}{a}\zeta_{4,m}(\alpha, 0) - \frac{Eh}{E_1F}\zeta_{3,m}(\alpha, 0)\right] + D_m s_m^2\left[\frac{\nu m\pi}{a}\zeta_{0,m}(\alpha, 0) - \right.\right.$$

$$- \frac{Eh}{E_1F}\zeta_{1,m}(\alpha, 0)\Big] - \frac{Eh}{E_1F}\sum_n \frac{4\beta m^2}{\pi^2\Delta_{mn}}A_n + \frac{Eh}{E_1F}\sum_n \frac{4\beta^3 t_n^2}{\pi^4\Delta_{mn}}B_n -$$

$$\left.- \frac{8\nu k^2 i a}{\pi^2 m^2 s_m^2}\alpha^* + \frac{4\nu qa}{Dm^2\pi^2}\left[\frac{1}{s_m^2}\zeta_{0,m}(\alpha, 0)\right] + \frac{Eh}{E_1F}\cdot\frac{4qa^2}{Dm^3\pi^3}\zeta_{1,m}(\alpha, 0)\right\} = 0,$$

$$\operatorname{Re}\left\{-\sum_n \frac{4nm^3}{b\Delta_{mn}}A_n + \sum_n \frac{4n\beta^2 m}{b\pi^2\Delta_{mn}}B_n t_n^2 + \right.$$

$$+ C_m\left[-\frac{m\pi}{a}\zeta_{3,m}(\alpha, 0) + \frac{D}{C}\zeta_{4,m}(\alpha, 0)\right] - D_m s_m^2\frac{m\pi}{a}\zeta_{1,m}(\alpha, 0) -$$

$$\left.- \frac{D}{C}\beta^*\cdot\frac{4}{m\pi} + \frac{4qa}{Dm^2\pi^3}\zeta_{1,m}(\alpha, 0)\right\} = 0,$$

$$\operatorname{Re}\left\{\left(\frac{a}{m\pi}\right)^3 D_m s_m^2\zeta_{0,m}(\alpha, 0) + \frac{2i\alpha^*}{\pi k^2 m s_m^2} + \frac{2i\beta^*}{\pi k^2 m} + \right.$$

$$\left.+ \frac{4qa^4}{Dm^5\pi^5}\left[\frac{1}{s_m^2} - \zeta_{0,m}(\alpha, 0)\right]\right\} = 0 \qquad (m = 1, 3, 5, \ldots),$$

where

$$\zeta_{1,n}(\beta, 0) = \frac{1}{1-t_n^2}\left[\operatorname{th}\frac{\beta}{2} - \frac{1}{t_n}\operatorname{th}\frac{t_n\beta}{2}\right],$$

$$\zeta_{3,n}(\beta, 0) = \frac{1}{1-t_n^2}\left[\operatorname{th}\frac{\beta}{2} - t_n\operatorname{th}\frac{t_n\beta}{2}\right],$$

$$\zeta_{1,m}(\alpha, 0) = \frac{1}{1-s_m^2}\left[\operatorname{th}\frac{\alpha}{2} - \frac{1}{s_m}\operatorname{th}\frac{s_m\alpha}{2}\right],$$

$$\zeta_{3,m}(\alpha, 0) = \frac{1}{1-s_m^2}\left[\operatorname{th}\frac{\alpha}{2} - s_m\operatorname{th}\frac{s_m\alpha}{2}\right],$$

$$\left.\begin{aligned}\zeta_{p+4,n}(\beta, 0) &= (1+t_n^2)\zeta_{p+2,n}(\beta, 0) + t_n^2\zeta_{p,n}(\beta, 0)\\ \zeta_{p+4,m}(\alpha, 0) &= (1+s_m^2)\zeta_{p+2,m}(\alpha, 0) + s_m^2\zeta_{p,m}(\alpha, 0)\end{aligned}\right\}(p=0, 1),$$

$$\Delta_{mn} = \left(m^2 + \frac{n^2a^2}{b^2}\right)^2 + 2i\frac{k^2a^2}{\pi^2b^2}(m^2b^2 + n^2a^2),$$

$$\delta_{mn} = \left(n^2 + \frac{m^2b^2}{a^2}\right)^2 + 2i\frac{k^2b^2}{\pi^2a^2}(n^2a^2 + m^2b^2).$$

The coefficients A_n, B_n, C_m, D_m can be found approximately by solving this system by the reduction method.

Example of an approximative determination of the moments and forces

Retaining only two terms in each series (7.105), we obtain a system of 16 equations for determining 16 unknown real coefficients. We analyzed in this way a shell whose projection in plan was a rectangle. The edges were reinforced by a metal welt. The following data were used:

$$a = 37\text{ cm},\quad b = 27\text{ cm},\quad h = 0.45\text{ cm},\quad R = 76\text{ cm},\quad \nu = 0.23,$$
$$E = 0.6946\cdot 10^6\text{ dyn.cm}^2\quad E_1 = 2\cdot 10^6\text{ dyn.cm}^2\quad q = 1\text{ dyn.cm}^2.$$

The stiffening rib shown in section in Figure 106 had the following rigidities in tension, torsion, and bending in two planes:

$E_1F = 2.4932\cdot 10^6$ dyn, $C = 322404$ dyn $\cdot$ cm^2, $\beta' = 878508$ dyn $\cdot$ cm^2, $\beta = 805941$ dyn$\cdot$cm^2.

The bending moments M_x, M_y and the longitudinal forces N_x, N_y acting along the axes of symmetry $\eta = 0.5$ and $\xi = 0.5$, have been plotted in Figures 107 to 109.

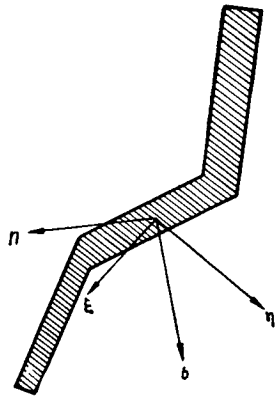


FIGURE 106.

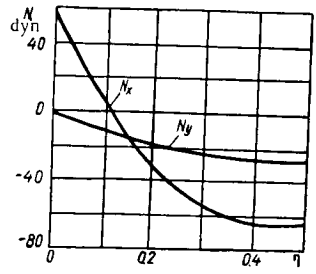


FIGURE 107.

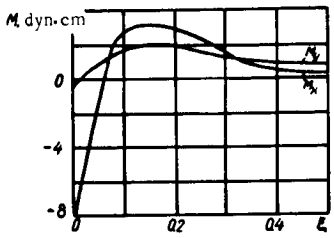


FIGURE 108.

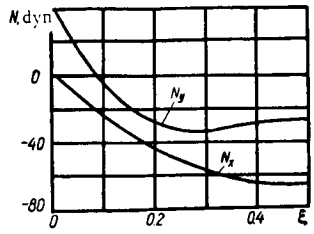


FIGURE 109.

It is seen that in the case considered the forces normal to the edge (N_x at the ends of the large axis $\eta = 0.5$ and N_y at the ends of the small axis $\xi = 0.5$) are negligibly small, namely

$$N_x = -0.198 \text{ dyn} \cdot \text{cm} \quad \text{at} \quad \begin{cases} \xi = 0, \\ \eta = 0.5; \end{cases}$$

$$N_y = -0.927 \text{ dyn} \cdot \text{cm} \quad \text{at} \quad \begin{cases} \xi = 0.5, \\ \eta = 0. \end{cases}$$

On the other hand, the normal forces acting at the same points on areas normal to the contour are considerable:

$$N_y = 41.61 \text{ dyn} \cdot \text{cm} \quad \text{at} \quad \begin{cases} \xi = 0, \\ \eta = 0.5; \end{cases}$$

$$N_x = 53.87 \text{ dyn} \cdot \text{cm} \quad \text{at} \quad \begin{cases} \xi = 0.5, \\ \eta = 0. \end{cases}$$

The bending moments M_x and M_y are small at the shell center, being largest at the edge.*

A comparison of these moment and force diagrams with those for a nonreinforced shell (cf. /41, 42/) shows that the reinforcement affects considerably the distribution of moments and forces in the shell. By suitably selecting the cross-sectional shape and dimensions of the stiffening ribs we can obtain an optimum stress distribution in the shell; this necessitates, however, very involved calculations.

§ 7. OTHER PROBLEMS CONCERNING THE REINFORCEMENT OF SHELLS ALONG THE EDGES OF HOLES

Few works have been written on the stress distribution and holes in thin shells. A survey of papers published on this subject up to the end of 1961 was given by G.N. Savin (/119/ to /121/). A.N. Guz' (/35, 36/), recently obtained approximate solutions for the stresses in a thin shell having an elliptic, triangular, or square hole (with rounded edges).

Using the method of "boundary perturbation," Guz' solved the problem of the elastic equilibrium of a shallow shell, whose projection in plan forms a circle, having a central elliptic, square, or triangular hole /37/.

V.G. Buivol /7/ solved, by the method of successive approximations, the problem of the stresses in uniformly loaded shallow spherical shell having four equal cyclical-symmetrically placed circular holes.

* Calculations by A.S. Gupalo and Ye.I. Lun', Department of Mechanics of L'vov State University.

The stress concentrations around holes in shells are usually reduced by reinforcing the hole edges by elastic elements. Yu. A. Shevlyakov /177/, M. M. Pirogov /99, 100/, and A. N. Guz' /38/ considered the influence of reinforcing elements like rigid and elastic washers on the stress concentrations around circular holes in spherical and cylindrical shells.

Boundary conditions (7.23) can be used to solve problems on the stresses in thin shells having holes whose edges are reinforced by thin elastic ribs.

Inverse problems similar to those solved in Chapter V were discussed by Mossakovskii and Kvash /80/ (cf. also /49, 50/), who considered the neutral reinforcement of hole edges in symmetrically loaded shells of revolution subjected only to membrane stresses, and in a spherical shell subjected to bending.

V. V. Vasil'ev (cf. "Prikladna Mekhanika," Vol. 7, Nos. 3, 4. 1961) investigated the axisymmetrical elastic-plastic state near the edge, reinforced by a rib, of a circular hole in a shell of revolution.

Papers /21, 22, 24, 113, 202/ are devoted to the stresses in a cylindrical framework shell having cuts extending over an integral number of elements.

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EXPLANATORY LIST OF ABBREVIATED NAMES OF USSR INSTITUTIONS, ORGANIZATIONS, JOURNALS, ETC., APPEARING IN THIS TEXT

Abbreviation	Full name (transliterated)	Translation
AN SSSR	Akademiya Nauk SSSR	Academy of Sciences of the USSR
BNI	Byulleten' Tekhnicheskoi Informatsii	Bulletin of Technical Information
DGU	Dnepropetrovskii Gosudarstvennyi Universitet	Dnepropetrovsk State University
KGU	Kievskii Gosudarstvennyi Universitet	Kiev State University
LDU	Leningradskii Dom Uchenykh (im. A. M. Gor'kogo)	Leningrad Scientists' House (im. A. M. Gor'kii)
MGU	Moskovskii Gosudarstvennyi Universitet	Moscow State University
MIKhM	Moskovskii Institut Khimicheskogo Mashinostroeniya	Moscow Institute of Chemical Machine Building
MVTU	Moskovskoe Vysshee Tekhnicheskoe Uchilishche (im. N. E. Baumana)	Moscow Higher Technical School (im. N. E. Bauman)
NKTP	Narodnyi Komissariat Tyazheloi Promyshlennosti	People's Commissariat of Heavy Industry
NTIU	Nauchno-Tekhnicheskoe Inzhenernoe Upravlenie	Scientific and Technical Engineering Administration
OTN	Otdelenie Tekhnicheskikh Nauk (Akademii Nauk SSSR)	Department of Technical Sciences (of the Academy of Sciences of the USSR)
PMM	Prikladnaya Matematika i Mekhanika	Applied Mathematics and Mechanics
TsAGI	Tsentral'nyi Aero-Gidrodinamicheskii Institut (im. N. E. Zhukovskogo)	Central Aero-Hydrodynamical Institute (im. N. E. Zhukovskii)
VINITI	Vsesoyuznyi Institut Nauchnoi i Tekhnicheskoi Informatsii	All-Union Institute for Scientific and Technical Information